6.4 Exercises

Exercise 6.1 Consider an experiment with three possible outcomes and odds as follows.

Outcome	Odds
1	1
2	2
3	5

Is there a betting scheme that results in a sure win?

Exercise 6.2 Consider an experiment with four possible outcomes, and suppose that the quoted odds for the first three of these outcomes are as follows.

Outcome	Odds
1	2
2	3
3	4

What must be the odds against outcome 4 if there is to be no possible arbitrage when one is allowed to bet both for and against any of the outcomes?

Exercise 6.3 Repeat Exercise 6.1 when the odds are as follows.

Outcome	Odds
1	2
2	2
3	2

Exercise 6.4 Suppose, in Exercise 6.1, that one may also choose any pair of outcomes $i \neq j$ and bet that the outcome will be either i or j.

What should the odds be on these three bets if an arbitrage opportunity is to be avoided?

Exercise 6.5 In Example 6.1a, show that if

$$\sum_{i=1}^{m} \frac{1}{1+o_i} \neq 1$$

then the betting scheme

$$x_i = \frac{(1+o_i)^{-1}}{1-\sum_{i=1}^{m}(1+o_i)^{-1}}, \quad i = 1, \ldots, m,$$

will always yield a gain of exactly 1.

Exercise 6.6 In Example 6.1b, suppose one also has the option of purchasing a put option that allows its holder to put the stock for sale at the end of one period for a price of 150. Determine the value of P, the cost of the put, if there is to be no arbitrage; then show that the resulting call and put prices satisfy the put–call option parity formula (Proposition 5.2.2).

Exercise 6.7 Suppose that, in each period, the cost of a security either goes up by a factor of 2 or goes down by a factor of $1/2$ (i.e., $u = 2$, $d = 1/2$). If the initial price of the security is 100, determine the no-arbitrage cost of a call option to purchase the security at the end of two periods for a price of 150.

Exercise 6.8 Suppose, in Example 6.1b, that there are three possible prices for the security at time 1: 50, 100, or 200. (That is, allow for the possibility that the security's price remains unchanged.) Use the arbitrage theorem to find an interval for which there is no arbitrage if C lies in that interval.

A betting strategy **x** such that (using the notation of Section 6.1)

$$\sum_{i=1}^{n} x_i r_i(j) \geq 0, \quad j = 1, \ldots, m,$$

with strict inequality for at least one j, is said to be a *weak arbitrage strategy*. That is, whereas an arbitrage is present if there is a strategy that results in a positive gain for every outcome, a weak arbitrage is present if there is a strategy that never results in a loss and results in a positive gain for at least one outcome. (An arbitrage can be thought of as a *free lunch*, whereas a weak arbitrage is a *free lottery ticket*.) It can be shown that there will be no weak arbitrage if and only if there is a probability vector **p**, all of whose components are positive, such that

$$\sum_{j=1}^{m} p_j r_i(j) = 0, \quad i = 1, \ldots, n.$$

In other words, there will be no weak arbitrage if there is a probability vector that gives positive weight to each possible outcome and makes all bets fair.

Exercise 6.9 In Exercise 6.8, show that a weak arbitrage is possible if the cost of the option is equal to either endpoint of the interval determined.

Exercise 6.10 For the model of Section 6.2 with $n = 1$, show how an option can be replicated by a combination of borrowing and buying the security.

Exercise 6.11 The price of a security in each time period is its price in the previous time period multiplied either by $u = 1.25$ or by $d = .8$. The initial price of the security is 100. Consider the following "exotic" European call option that expires after five periods and has a strike price of 100. What makes this option exotic is that it becomes alive only if the price after two periods is strictly less than 100. That is, it becomes alive only if the price decreases in the first two periods. The final payoff of this option is

$$\text{payoff at time } 5 = I(S(5) - 100)^+,$$

where $I = 1$ if $S(2) < 100$ and $I = 0$ if $S(2) \geq 100$. Suppose the interest rate per period is $r = .1$.

(a) What is the no-arbitrage cost (at time 0) of this option?
(b) Is the cost of part (a) unique? Briefly explain.
(c) If each price change is equally likely to be an up or a down movement, what is the expected amount that an option holder receives at the time of expiration?

REFERENCES

[1] De Finetti, Bruno (1937). "La prevision: ses lois logiques, ses sources subjectives." *Annales de l'Institut Henri Poincaré* 7: 1–68; English translation in S. Kyburg (Ed.) (1962), *Studies in Subjective Probability*, pp. 93–158. New York: Wiley.
[2] Gale, David (1960). *The Theory of Linear Economic Models*. New York: McGraw-Hill.