call option on security j refer to a call option having strike price K_j and expiration time t, and let C_j $(j = 1, \ldots, n)$ denote the costs of these options. Also, let C be the cost of a call option on the index I that has strike price $\sum_{j=1}^{n} w_j K_j$ and expiration time t. We now show that the payoff of the call option on the index is always less than or equal to the sum of the payoffs from buying w_j (K_j, t) call options on security j for each $j = 1, \ldots, n$:

$$
\begin{aligned}
&\text{index option payoff at time } t \\
&= \left(I(t) - \sum_{j=1}^{n} w_j K_j\right)^+ \\
&= \left(\sum_{j=1}^{n} w_j S_j(t) - \sum_{j=1}^{n} w_j K_j\right)^+ \\
&= \left(\sum_{j=1}^{n} w_j (S_j(t) - K_j)\right)^+ \\
&\le \left(\sum_{j=1}^{n} (w_j (S_j(t) - K_j))^+\right)^+ \qquad (\text{because } x \le x^+) \\
&= \left(\sum_{j=1}^{n} w_j (S_j(t) - K_j)^+\right)^+ \\
&= \sum_{j=1}^{n} w_j (S_j(t) - K_j)^+ \\
&= \sum_{j=1}^{n} w_j \cdot [\text{payoff from } (K_j, t) \text{ call option}].
\end{aligned}
$$

Consequently, by the generalized law of one price, we have that either $C \le \sum_{j=1}^{n} w_j C_j$ or there is an arbitrage. □

5.3 Exercises

Exercise 5.1 Suppose it is known that the price of a certain security after one period will be one of the r values $s_1, \ldots, s_r$. What should be

the cost of an o
when $K <$ min

Exercise 5.2
whose present p

Exercise 5.3
rity at time t fo
and let r be the
quantities C, S

Exercise 5.4
rate of 5%, con
of a call option

Exercise 5.5
present price is
sarily true?

(a) $P \le S$.
(b) $P \le K$.

Exercise 5.6
present price is

where t is the

Exercise 5.7
of selling one s
option always

Exercise 5.8
ity formula.

Exercise 5.9
expire in three
the price 3. I
10% and the s

the cost of an option to purchase the security at time 1 for the price K when $K < \min s_i$?

Exercise 5.2 Let C be the price of a call option to purchase a security whose present price is S. Argue that $C \leq S$.

Exercise 5.3 Let C be the cost of a call option to purchase a security at time t for the price K. Let S be the current price of the security, and let r be the interest rate. State and prove an inequality involving the quantities C, S, and Ke^{-rt}.

Exercise 5.4 The current price of a security is 28. Given an interest rate of 5%, compounded continuously, find a lower bound for the price of a call option that expires in four months and has a strike price of 30.

Exercise 5.5 Let P be the price of a put option to sell a security, whose present price is S, for the amount K. Which of the following are necessarily true?

(a) $P \leq S$.
(b) $P \leq K$.

Exercise 5.6 Let P be the price of a put option to sell a security, whose present price is S, for the amount K. Argue that

$$P \geq Ke^{-rt} - S,$$

where t is the exercise time and r is the interest rate.

Exercise 5.7 With regard to Proposition 5.2.2, verify that the strategy of selling one share of stock, selling one put option, and buying one call option always results in a positive win if $S + P - C > Ke^{-rt}$.

Exercise 5.8 Use the law of one price to prove the put–call option parity formula.

Exercise 5.9 A European call and put option on the same security both expire in three months, both have a strike price of 20, and both sell for the price 3. If the nominal continuously compounded interest rate is 10% and the stock price is currently 25, identify an arbitrage.

Exercise 5.10 Let C_a and P_a be the costs of American call and put options (respectively) on the same security, both having the same strike price K and exercise time t. If S is the present price of the security, give either an identity or an inequality that relates the quantities C_a, P_a, K, and e^{-rt}. Briefly explain.

Exercise 5.11 Consider two put options on the same security, both of which have expiration t. Suppose the exercise prices of the two puts are K_1 and K_2. Argue that

$$K_1 - K_2 \geq P_1 - P_2,$$

where P_i is the price of the put with strike K_i, $i = 1, 2$.

Exercise 5.12 Explain why the price of an American put option having exercise time t cannot be less than the price of a second put option on the same security that is identical to the first option except that its exercise time is earlier.

Exercise 5.13 Say whether each of the following statements is always true, always false, or sometimes true and sometimes false. Assume that, aside from what is mentioned, all other parameters remain fixed. Give brief explanations for your answers.

(a) The price of a European call option is nondecreasing in its expiration time.
(b) The price of a forward contract on a foreign currency is nondecreasing in its maturity date.
(c) The price of a European put option is nondecreasing in its expiration time.

Exercise 5.14 Your financial adviser has suggested that you buy both a European put and a European call on the same security, with both options expiring in three months, and both having a strike price equal to the present price of the security.

(a) Under what conditions would such an investment strategy seem reasonable?
(b) Plot the return at time $t = 1/4$ from this strategy as a function of the price of the security at that time.

Exercise 5.15 If a stock is selling for a price s immediately before it pays a dividend d (i.e., the amount d per share is paid to every shareholder), then what should its price be immediately after the dividend is paid?

Exercise 5.16 Let $S(t)$ be the price of a given security at time t. All of the following options have exercise time t and, unless stated otherwise, exercise price K. Give the payoff at time t that is earned by an investor who:

(a) owns one call and one put option;
(b) owns one call having exercise price K_1 and has sold one put having exercise price K_2;
(c) owns two calls and has sold short one share of the security;
(d) owns one share of the security and has sold one call.

Exercise 5.17 Argue that the price of a European call option is nonincreasing in its strike price.

Exercise 5.18 Suppose that you simultaneously buy a call option with strike price 100 and write (i.e., sell) a call option with strike price 105 on the same security, with both options having the same expiration time.

(a) Is your initial cost positive or negative?
(b) Plot your return at expiration time as a function of the price of the security at that time.

Exercise 5.19 Consider two call options on a security whose present price is 110. Suppose that both call options have the same expiration time; one has strike price 100 and costs 20, whereas the other has strike price 110 and costs C. Assuming that an arbitrage is not possible, give a lower bound on C.

Exercise 5.20 Let $P(K, t)$ denote the cost of a European put option with strike K and expiration time t. Prove that $P(K, t)$ is convex in K for fixed t, or explain why it is not necessarily true.

Exercise 5.21 Can the proof given in the text for the cost of a call option be modified to show that the cost of an American put option is convex in its strike price?

Exercise 5.22 A (K_1, t_1, K_2, t_2) *double call option* is one that can be exercised either at time t_1 with strike price K_1 or at time t_2 $(t_2 > t_1)$ with strike price K_2. Argue that you would never exercise at time t_1 if $K_1 > e^{-r(t_2-t_1)}K_2$.

Exercise 5.23 In a *capped call option*, the return is capped at a certain specified value A. That is, if the option has strike price K and expiration time t, then the payoff at time t is

$$\min(A, (S(t) - K)^+),$$

where $S(t)$ is the price of the security at time t. Show that an equivalent way of defining such an option is to let

$$\max(K, S(t) - A)$$

be the strike price when the call is exercised at time t.

Exercise 5.24 Argue that an American capped call option should be exercised early only when the price of the security is at least $K + A$.

Exercise 5.25 A function $f(x)$ is said to be *concave* if, for all x, y and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y).$$

(a) Give a geometrical interpretation of when a function is concave.
(b) Argue that $f(x)$ is concave if and only if $g(x) = -f(x)$ is convex.

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