Solution. Rewriting $r(s)$ as

$$r(s) = r_2 + \frac{r_1 - r_2}{1+s}, \quad s \geq 0,$$

shows that the yield curve is given by

$$\begin{aligned}\bar{r}(t) &= \frac{1}{t}\int_0^t \left(r_2 + \frac{r_1 - r_2}{1+s}\right) ds \\ &= r_2 + \frac{r_1 - r_2}{t}\log(1+t).\end{aligned}$$

Consequently, the present value function is

$$\begin{aligned}P(t) &= \exp\{-t\bar{r}(t)\} \\ &= \exp\{-r_2 t\}\exp\{-\log((1+t)^{r_1 - r_2})\} \\ &= \exp\{-r_2 t\}(1+t)^{r_2 - r_1}.\end{aligned}$$

□

4.5 Exercises

Exercise 4.1 What is the effective interest rate when the nominal interest rate of 10% is

(a) compounded semiannually;
(b) compounded quarterly;
(c) compounded continuously?

Exercise 4.2 Suppose that you deposit your money in a bank that pays interest at a nominal rate of 10% per year. How long will it take for your money to double if the interest is compounded continuously?

Exercise 4.3 If you receive 5% interest compounded yearly, approximately how many years will it take for your money to quadruple? What if you were earning only 4%?

Exercise 4.4 Give a formula that approximates the number of years it would take for your funds to triple if you received interest at a rate r compounded yearly.

Exercise 4.5 How much do you need to invest at the beginning of each of the next 60 months in order to have a value of \$100,000 at the end of 60 months, given that the annual nominal interest rate will be fixed at 6% and will be compounded monthly?

Exercise 4.6 The yearly cash flows of an investment are

$$-1{,}000, \quad -1{,}200, \quad 800, \quad 900, \quad 800.$$

Is this a worthwhile investment for someone who can both borrow and save money at the yearly interest rate of 6%?

Exercise 4.7 Consider two possible sequences of end-of-year returns:

$$20, \; 20, \; 20, \; 15, \; 10, \; 5 \quad \text{and} \quad 10, \; 10, \; 15, \; 20, \; 20, \; 20.$$

Which sequence is preferable if the interest rate, compounded annually, is: (a) 3%; (b) 5%; (c) 10%?

Exercise 4.8 A five-year \$10,000 bond with a 10% coupon rate costs \$10,000 and pays its holder \$500 every six months for five years, with a final additional payment of \$10,000 made at the end of those ten payments. Find its present value if the interest rate is: (a) 6%; (b) 10%; (c) 12%. Assume the compounding is monthly.

Exercise 4.9 A friend purchased a new sound system that was selling for \$4,200. He agreed to make a down payment of \$1,000 and to make 24 monthly payments of \$160, beginning one month from the time of purchase. What is the effective interest rate being paid?

Exercise 4.10 Repeat Example 4.2b, this time assuming that the yearly interest rate is 20%.

Exercise 4.11 Repeat Example 4.2b, this time assuming that the cost of a new machine increases by \$1,000 each year.

Exercise 4.12 Suppose you have agreed to a bank loan of \$120,000, for which the bank charges no fees but 2 points. The quoted interest rate is .5% per month. You are required to pay only the accumulated interest each month for the next 36 months, at which point you must make a balloon payment of the still-owed \$120,000. What is the effective interest rate of this loan?

Exercise 4.13 You can pay off a loan either by paying the entire amount of \$16,000 now or you can pay \$10,000 now and \$10,000 at the end of ten years. Which is preferable when the nominal continuously compounded interest rate is: (a) 2%; (b) 5%; (c) 10%?

Exercise 4.14 A U.S. treasury bond (selling at a *par value* of \$1,000) that matures at the end of five years is said to have a *coupon rate* of 6% if, after paying \$1,000, the purchaser receives \$30 at the end of each of the following nine six-month periods and then receives \$1,030 at the end of the the tenth period. That is, the bond pays a simple interest rate of 3% per six-month period, with the principal repaid at the end of five years. Assuming a continuously compounded interest rate of 5%, find the present value of such a stream of cash payments.

Exercise 4.15 Explain why it is reasonable to suppose that $(1+.05/n)^n$ is an increasing function of n for $n = 1, 2, 3, \dots$.

Exercise 4.16 A bank pays a nominal interest rate of 6%, continuously compounded. If 100 is initially deposited, how much interest will be earned after

(a) 30 days;
(b) 60 days;
(c) 120 days?

Exercise 4.17 Assume continuously compounded interest at rate r. You plan to borrow 1,000 today, 2,000 one year from today, 3,000 two years from today, and then pay off all these loans three years from today. How much will you have to pay?

Exercise 4.18 The nominal interest rate is 5%, compounded yearly. How much would you have to pay today in order to receive the string of payments 3, 5, −6, 5, where the ith payment is to be received i years from now, $i = 1, 2, 3, 4$. (The payment −6 means that you will have to pay 6 three years from now.)

Exercise 4.19 Let r be the nominal interest rate, compounded yearly. For what values of r is the cash flow stream 20, 10 preferable to the cash flow stream 0, 34?

Exercise 4.20 Determine the length of time necessary for a bank deposit of 1,000 to grow to 1,500 if the nominal continuously compounded interest rate is 6%.

Exercise 4.21 Assuming continuously compounded interest at rate r, what is the present value of a cash flow sequence that returns the amount A at each of the times $s, s + t, s + 2t, \dots$?

Exercise 4.22 Let $D(t)$ denote the amount you would have on deposit at time t if you deposit D at time 0 and interest is continuously compounded at rate r.

(a) Argue that, for h small, $D(t + h) \approx D(t) + rhD(t)$.
(b) Use (a) to argue that $D'(t) = rD(t)$.
(c) Use (b) to conclude that $D(t) = De^{rt}$.

Exercise 4.23 Consider two cash flow streams, where each will return the ith payment after i years:

$$100, 140, 131 \quad \text{and} \quad 90, 160, 120.$$

Is it possible to tell which cash flow stream is preferable without knowing the interest rate?

Exercise 4.24 For an initial investment of 20, you will receive after one period a return that will equal either 10 with probability .3 or 40 with probability .7. What is the expected value of the rate of return for this investment?

Exercise 4.25 A zero coupon rate bond having face value F pays the bondholder the amount F when the bond matures. Assuming a continuously compounded interest rate of 8%, find the present value of a zero

coupon bond with
years.

Exercise 4.26 F
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Exercise 4.27
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Exercise 4.28
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Exercise 4.29
c_n, where c_i

coupon bond with face value $F = 1{,}000$ that matures at the end of ten years.

Exercise 4.26 Find the rate of return of a two-year investment that, for an initial payment of 1,000, gives a return at the end of the first year of 500 and a return at the end of the second year of: (a) 300; (b) 500; (c) 700.

Exercise 4.27 Repeat the preceding exercise, reversing the order in which the payments are received.

Exercise 4.28 The inflation rate is defined to be the rate at which prices as a whole are increasing. For instance, if the yearly inflation rate is 4% then what cost \$100 last year will cost \$104 this year. Let r_i denote the inflation rate, and consider an investment whose rate of return is r. We are often interested in determining the investment's rate of return from the point of view of how much the investment increases one's purchasing power; we call this quantity the investment's *inflation-adjusted rate of return* and denote it as r_a. Since the purchasing power of the amount $(1+r)x$ one year from now is equivalent to that of the amount $(1+r)x/(1+r_i)$ today, it follows that – with respect to constant purchasing power units – the investment transforms (in one time period) the amount x into the amount $(1+r)x/(1+r_i)$. Consequently, its inflation-adjusted rate of return is

$$r_a = \frac{1+r}{1+r_i} - 1.$$

When r and r_i are both small, we have the following approximation:

$$r_a \approx r - r_i.$$

For instance, if a bank pays a simple interest rate of 5% when the inflation rate is 3%, the inflation-adjusted interest rate is approximately 2%. What is its exact value?

Exercise 4.29 Consider an investment cash flow sequence $c_0, c_1, \dots, c_n$, where $c_i < 0$, $i < n$, and $c_n > 0$. Show that if

$$P(r) = \sum_{i=0}^{n} c_i(1+r)^{-i}$$

then, in the region $r > -1$,

(a) there is a unique solution of $P(r) = 0$;
(b) $P(r)$ need not be a monotone function of r.

Exercise 4.30 Suppose you can borrow money at an annual interest rate of 8% but can save money at an annual interest rate of only 5%. If you start with zero capital and if the yearly cash flows of an investment are

$$-1{,}000,\ 900,\ 800,\ -1{,}200,\ 700,$$

should you invest?

Exercise 4.31 Show that, if $r(t)$ is an nondecreasing function of t, then so is $\bar{r}(t)$.

Exercise 4.32 Show that the yield curve $\bar{r}(t)$ is a nondecreasing function of t if and only if

$$P(\alpha t) \geq (P(t))^{\alpha} \quad \text{for all } 0 \leq \alpha \leq 1,\ t \geq 0.$$

Exercise 4.33 If $P(t) = e^{-a-bt}$ $(t \geq 0)$, find: (a) $r(t)$; (b) $\bar{r}(t)$.

Exercise 4.34 Show that

$$\text{(a)}\ r(t) = -\frac{P'(t)}{P(t)} \quad \text{and} \quad \text{(b)}\ \bar{r}(t) = -\frac{\log P(t)}{t}.$$

Exercise 4.35 Plot the spot interest rate function $r(t)$ of Example 4.4a when

(a) $r_1 < r_2$;
(b) $r_2 < r_1$.