§2

2.1 ① $P(Z<-0.66) = P(Z>0.66) = 1-P(Z\le 0.66) = 1-\phi(0.66)$

② $P(|Z|<1.64) = P(Z<1.64) - P(Z<-1.64)$

$= P(Z<1.64) - (1-P(Z<1.64))$

$= 2P(Z<1.64) - 1$

③ $P(|Z|>2.20) = 2P(Z>2.20) = 2(1-P(Z<2.20))$

$= 2(1-0.9861) = 0.0278$

2.2.

$P(-2<Z<1) = P(1<Z<2)$ by symmetry

2.3. $P(|Z|>x) = P(Z>x) + P(Z<-x) = 2P(Z>x)$

2.4. ~~$Cov(X, Y) = Cov(X, aX+bY)$~~

$E(Y) = a + b\,E(x) = \mu = a + b\mu$

$Var(Y) = b^2\sigma = \sigma \Rightarrow b = \pm 1$

$b=1 \Rightarrow 0=a$ $\qquad$ $\underline{b=-1 \quad a=2\mu}$

$\times$

$Cov(X, Y) = Cov(X, 2\mu\cancel{0} - X)$

$= -Var(X) = -\sigma$

2.6. $X_1 . X_2$ the life of the 1st and 2nd battery life

ⓐ $P(X_1+X_2>760) = P\left(\frac{X_1+X_2-800}{50\sqrt{2}} > \frac{760-800}{50\sqrt{2}}\right)$

$= P(Z>-0.5657) = P(Z<0.5657)$

$= \phi(0.5657) = 0.7142$

ⓑ $P(X_2-X_1>25) = P\left(\frac{X_2-X_1-0}{50\sqrt{2}} > \frac{25-0}{50\sqrt{2}}\right)$

$E(X_2-X_1)=0$

$Var(X_1 \mp X_2) = Var(X_1) + Var(X_2)$

$= P(Z>0.3536) = 1-\phi(0.3536)$

$= 1-0.6382 = 0.3618$

ⓒ $P(|X_1-X_2|>25) = P(X_2-X_1>25) = 0.3618 * 2$

$= 0.7236$

2-7. $X_i \to$ time for i-th phote then $X = \sum_{i=1}^{100} X_i$ $E(X)=1800$ $Var(X)=100$

ⓐ $P(X>1710) = \left(\frac{X-1800}{\sqrt{100}} > \frac{1710-1800}{\sqrt{100}}\right)$

$= P(Z>-9) = P(Z<9) = \Phi(9) \approx 1$

ⓑ $P(\cancel{X} 1690 < X < 1710) = P\left(\frac{1690-1800}{\sqrt{100}} < \frac{X-1800}{\sqrt{100}} < \frac{1710-1800}{\sqrt{100}}\right)$

$= P(-11 < Z < -9) = P(9 < Z < 11)$

$= P(Z<11) - (1-P(Z<9))$

$= \phi(11) + \Phi(9) - 1$

2.8. let X_i : mileage for i person

$$E[X_i] = 25000$$

$$Var(X_i) = 12000^2$$

$$X = \sum_{i=1}^{30} X_i / 30 \qquad \cancel{E[X] = \frac{1}{30} * 30 * 25000 = 25000}$$

ⓐ

$$P(X > 25000) = P\left(\frac{\sum_{i=1}^{30} X_i}{30} > 25000\right) = P\left(\sum_{i=1}^{30} X_i > 25000 * 30\right)$$

$$= P\left(\frac{\sum_{i=1}^{30} X_i - 25000 * 30}{12000 \cdot \sqrt{30}} > 0\right)$$

$$= \phi(0) = 0.5$$

ⓑ

$$P(23000 * 30 < \sum X_i < 27000 * 30)$$

$$= P\left(\frac{23000 * 30 - 25000 * 30}{12000\sqrt{30}} < \frac{\sum X_i - 25000 * 30}{12000\sqrt{30}} < \frac{27000 * 30 - 25000 * 30}{12000\sqrt{30}}\right)$$

$$= P\left(\frac{-5}{\sqrt{30}} < Z < \frac{5}{\sqrt{30}}\right)$$

$$= \phi\left(\frac{5}{\sqrt{30}}\right) - \phi\left(-\frac{5}{\sqrt{30}}\right) = 2\phi\left(\frac{5}{\sqrt{30}}\right) - 1 = 0.6388$$

2.9 let S_i be the price of time i

$$S_{i+1} = S_i X_i \quad \text{where} \quad X_i = \begin{cases} u & @\ p \\ d & @\ 1-p \end{cases}$$

$$P\left(\frac{S_{1000}}{S_0} > 1.3\right) = P\left(\frac{S_{1000}}{S_{999}} \cdot \frac{S_{999}}{S_{998}} \cdots \frac{S_2}{S_1}\frac{S_1}{S_0} > 1.3\right)$$

$$= P(X_{999} \cdot X_{998} \cdot \cdots \cdot X_0 > 1.3)$$

$$= P\left(\sum_{i=0}^{999} \log X_i > \log 1.3\right)$$

let $Y = \sum_{i=0}^{999} \log X_i$

$$E[Y] = \sum_{i=0}^{999} E[\log X_i] = 1000\left(p \log u + (1-p)\log d\right) = 1.3787$$

$$Var(Y) = 1000\, Var(\log X_i)$$

$$= 1000\left(E[(\log X_i)^2] - E^2[\log X_i]\right)$$

$$= 1000 \cdot \left(\log^2 u \cdot p + \log^2 d\,(1-p) - 0.0013787^2\right)$$

$$= 0.1206$$

so

$$P\left(\sum_{i=0}^{999} \log X_i > \log 1.3\right) = P(Y > \log 1.3)$$

$$= P\left(\frac{Y - 1.3787}{\sqrt{0.1206}} > \frac{\log 1.3 - 1.3787}{\sqrt{0.1206}}\right)$$

$$= P(Z > -3.2146) = \phi(3.2146) = 0.9993$$

2.10.

$$Y = \sum_{i=1}^{700} X_i \qquad X_i \text{ is the movement in period } i$$

$$E[Y] = E[\sum_{i=1}^{700} X_i] = 700(-0.39 + 0.41) = 700 * 0.02 = 14$$

$$Var(Y) = Var(\sum X_i) \cancel{= 700(0.39*}$$

$$= 700\, Var(X_i) = 700\, E([X_i - E(X_i)]^2)$$

$$(X_i - E[X_i])^2 = \begin{cases} 0.98^2 & @\ 0.41 \\ 0.02^2 & @\ 0.20 \\ 1.02^2 & @\ 0.39 \end{cases}$$

$$= 700 \cdot (0.98^2 * 0.41 + 0.02^2 * 0.2 + 1.02^2 * 0.39)$$

$$= 559.72.$$

so. $$P(\sum_{i=1}^{700} X_i > 10) = P(Y > 10) = P(\frac{Y-14}{\sqrt{559.72}} - \frac{10-14}{\sqrt{559.72}})$$

$$= P(Z > \frac{-4}{\sqrt{559.72}})$$

$$= P(Z < \frac{4}{\sqrt{559.72}}) = \phi(\frac{4}{\sqrt{559.72}})$$

§3

3.1 For $Y = -X(t)$

$Y(t+y) - Y(y) = -X(t+y) - (-X(y)) = -(X(t+y) - X(y))$

so $E[Y(t+y) - Y(y)] = -\mu t$

$Var(\qquad) = \cancel{4}\sigma^2 t$

3.2. ⓐ $E[X(2) - X(0)] = \cancel{6}\,3 \cdot 2 = 6 \Rightarrow E[X_2] = 16$

ⓑ $Var(X(2) - X(0)) = 2 \cdot 9 = 18 \Rightarrow Var(X_2) = 18$

ⓒ $P(X(2) > 20) = P\left(\frac{X(2) - 16}{\sqrt{18}} > \frac{20 - 16}{\sqrt{18}}\right)$

$= P(Z > 0.9428)$

ⓓ. $P(X(0.5) > 10)$

$E[X(0.5) - X(0)] = 1.5 \Rightarrow E[X(0.5)] = 11.5$

$Var(X(0.5) - X(0)) = 9 \cdot 0.5 = 4.5$

$P(X(0.5) > 10) = P\left(\frac{X(0.5) - 11.5}{\sqrt{4.5}} > \frac{10 - 11.5}{\sqrt{4.5}}\right)$

$= \ldots$

3.4 $S(t) = S(0)e^{W(t)}$ $W(0)=0$ $W(t)$ is Browinan Motion

(a) $P(S(1)>S(0)) = P(e^{W(1)}>1) = P(W(1)>0)$ $E(W(1)) = \mu \cdot 1$, $Var(W(1)) = \sigma^2 \cdot 1$

$= P\left(\frac{W(1)-0.1}{\sqrt{0.2^2}} > \frac{0-0.1}{\sqrt{0.2^2}}\right) = P(Z > -0.5)$

$= P(Z<0.5) = 0.6915$

(b) $P(S(2)>S(1)>S(0)) = P((S(2)>S(1)) \cap (S(1)>S(0)))$

$S(1)>S(0)$ is indepenelent with $S(2)>S(1)$

$= P(S(1)>S(0)) \cdot P(S(2)>S(1))$

$P(S(2)>S(1)) = P(S(0)e^{W(2)} > S(0)e^{W(1)}) = P(W(2)-W(1)>0)$

$E[W(2)-W(1)) = \mu \cdot 1$

$Var(W(2)-W(1)) = \sigma^2 \cdot 1$

so $P(S(2)-S(1)) = 0.6915$

so $P(S(2)>S(1)>S(0)) = (0.6915)^2$

(c) $P(S(3)< S(1) > S(0)) = P(S(1)>S(0))\, P(S(3) < S(1) \mid S(1)>S(0))$

independent

$= P(S(1)>S(0)) \cdot P(S(3) < S(1))$

$P(S(1)>S(0)) = 0.6915$

$P(S(3)/S(1) < 1) = P(e^{W(3)-W(1)} < 1) = P(W(3)-W(1) < 0)$

$E[W(3)-W(1)] = \mu \cdot 2$ $P\left(\frac{W(3)-W(1)-0.2}{\sqrt{2}\cdot 0.2} < \frac{-0.2}{\sqrt{2}\cdot 0.2}\right) = P\left(Z < \frac{-0.2}{\sqrt{0.08}}\right)$

$Var(W(3)-W(1)) = \sigma^2 \cdot 2$

so $P(S(3) \leq S(1) > S(0)) = 0.6915 \cdot P\left(Z < \frac{-0.2}{\sqrt{0.08}}\right)$

$$= 0.6915\left(1 - P\left(Z < \frac{0.2}{\sqrt{0.08}}\right)\right)$$

$$= 0.6915 \cdot \phi\left(\frac{0.2}{\sqrt{0.08}}\right)$$