A More Rigorous Approach to Limits

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Overview

The rigorous ϵ–δ definition of limits can be difficult for students to grasp. This lab is designed to provide visual and interactive tools for working with these concepts.

Maple Essentials

- The *EpsilonDelta* maplet is available from the course website:

  `http://www.math.sc.edu/calclab/141L-F07/labs/` → EpsilonDelta

Related course material/Preparation

§2.4 (Pages 134-143) of the textbook (Anton, 8th edition). Let us first recall the definition of limit given there: Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not be defined at a. We will write $\lim_{x\to a} f(x) = L$ if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - a| < \delta.$$

In general, ϵ and δ are meant to be very small numbers. Therefore, intuitively, the definition states that $f(x)$ will be very close to L that is, $|f(x) - L| < \epsilon$, when x is very close to a ($|x - a| < \delta$). The task is to show that, for any given ϵ (no matter how close $f(x)$ is to L), you can always find a δ–needed closeness of x to a–to make it work.

Activities

From our discussion, our job is to find a δ for a given ϵ such that, when $a-\delta < x < a+\delta$, the inequality $|f(x)-L| < \epsilon$ holds. Therefore, we need to solve for a range $a-\delta < x < a+\delta$ of x from the given inequality $|f(x)-L| < \epsilon$. Ideally, we would like to find a formula of δ in terms of ϵ (see examples 1, 2, and 3 of §2.4) that will work for any given ϵ. However, such formulas are in general very hard to find. Moreover, the value of δ is not unique, as any value that is smaller than a solution would work, too. For each of the limits below, we will use Maple's `solve` command to help us to find the largest δ that works for the given ϵ and the interactive *EpsilonDelta* maplet provides a tool to visualize relations between δ and ϵ.

(Follow the General Directions on the back of this page.)

1. $\lim_{x\to 9} \sqrt{x} = 3,\ \epsilon = 0.15,\ \epsilon = 0.05$
2. $\lim_{x\to 3} \dfrac{x^2 - 9}{x - 3} = 6,\ \epsilon = 0.2,\ \epsilon = 0.05$
3. $\lim_{x\to 3}(5x - 2) = 13,\ \epsilon = .05,\ \epsilon = .01$
4. $\lim_{x\to 2}(x^2 + 3x - 1) = 9,\ \epsilon = 0.8,\ \epsilon = 0.6$

   Hint: Since we also have $\lim_{x\to -5}(x^2 + 3x - 1) = 9$, Maple's `solve` command will return extra solutions. Which interval should you choose for problem 4?

Use Maple's `solve` command to solve inequalities

Maple's `solve` command was introduced in the Lab 4 to solve equations. It can also be used to solve inequalities. We will input most of our inequalities as follows:
> solve(abs(f(x)−L) < ϵ, x);
For example, if we want to know where $|\sqrt{x} - 2| < 0.05$ we would use the following command
> solve(abs(sqrt(x)−2) < 0.05, x);
and Maple would return the interval $(3.8025, 4.2025)$ as the solution (your TA will explain Maple's notation)

General Directions

1. Look at the limit and identify $f(x)$, L, a, and ϵ.
2. Launch the *EpsilonDelta* maplet and click **Modify or Make Your Own Problem**. Enter the function $f(x)$, a, L, and ϵ.
3. Click **Save Problem and Close**. You should see the graph of $f(x)$ in blue with a cyan vertical stripe that goes from $a - \delta$ to $a + \delta$ and a pink horizontal stripe that goes from $L - \epsilon$ to $L + \epsilon$. You should also see a brown rectangle extends vertically from the smallest value of $f(x)$ to the largest value of $f(x)$ for x from $a - \delta$ to $a + \delta$. You may change the size of this rectangle by changing the value of δ, which can be done using the slider (for $0.1 \leq \delta \leq 1$) or by typing in (any value).
4. Your task is to determine the largest value of δ that keeps the brown rectangle completely inside the pink stripe. You can use **Zoom In** to increase the accuracy.
5. When you think you are done, record your final value of δ.
6. Now we will find the value of δ more precisely using Maple's `solve` command.
7. Use the arrow notation (`:=x->`) to define the function $f(x)$. Use `:=` to assign L, a, and `epsilon` to their respective values.
8. Use the `solve` command as follows
   > solve(abs(f(x) − L) < epsilon, x);
   Maple should return an interval or intervals.
9. Choose the interval that contains a. Find the distances from a to the left bound and from a to the right bound of the interval (both of them should be positive.) The *smallest* of these two values is the *largest* δ that works for this ϵ.
10. Your values from the *EpsilonDelta* maplet and from using the `solve` command should be very close.

Remark:

For some simple functions like linear functions, `solve` can be used to find general formulas of δ in term of ϵ. Try the following and compare it to problem 3:

solve(abs(5*x-2-13)<epsilon,x) assuming epsilon>0;

Assignment

Exercises 9, 12, and 13 in §2.4 on page 141. Review Labs 1-5 for next week's Hour Quiz 1 (to be completed in lab).