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PARTITION VALUES AND CENTRAL CRITICAL VALUES OF CERTAIN MODULAR L-FUNCTIONS

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ABSTRACT. Let $p(n)$ denote the number of partitions of a positive number n, let $\ell \in \{5, 7, 11\}$ and let δ_ℓ be the least non-negative residue of 24^{-1} modulo ℓ. In this paper we prove congruences modulo ℓ between $\frac{p(\ell n+\delta_\ell)}{\ell}$ and ratios of central critical values of L-functions associated to twists of certain integer weight newforms. In 1999, Guo and Ono proved analogous results for $13 \leq \ell \leq 31$.

1. INTRODUCTION

Let $p(n)$ be the number of partitions of a positive integer n: the number of non-increasing sequences of positive integers whose sum is n. By convention, we agree that $p(0) := 1$. Partitions play an important role in number theory, combinatorics, representation theory, and Lie algebras. See, for example, [And98], [Ono04], [Ono08] and the references therein. The infinite product generating function of Euler,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} (1-q^m)^{-1}, \tag{1.1}$$

is the starting point for much of the modern study of $p(n)$. Using this formula, Ramanujan discovered and often proved interesting results on the arithmetic of $p(n)$. The most famous of these are the Ramanujan congruences, which we now describe. Let $\ell \geq 5$ be prime, and define

$$\delta_\ell := \ell\left(1 + \left\lfloor \frac{\ell}{24} \right\rfloor\right) - \left(\frac{\ell^2-1}{24}\right). \tag{1.2}$$

Note that δ_ℓ is the least non-negative residue of 24^{-1} modulo ℓ. Ramanujan proved that if $\ell \in \{5, 7, 11\}$, then for all $n \geq 0$ we have

$$p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell}. \tag{1.3}$$

These congruences have inspired terrific interest in the general study of linear congruences for Fourier coefficients of modular forms. Using the fact that (1.1) is closely related to a weakly holomorphic modular form of weight $-1/2$, Ahlgren and Ono in [Ahl00], [AO01], and [Ono00] proved that congruences for $p(n)$ are actually common in the following sense: for all $m \geq 1$ with $\gcd(m, 6) = 1$, there are

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infinitely many non-nested arithmetic progressions $An+B$ such that $p(An+B) \equiv 0 \pmod{m}$ for all non-negative integers n. On the other hand, Ahlgren and Boylan [AB03] proved that congruences of the type that Ramanujan found are quite rare; to be precise, the only (ℓ, d) with ℓ prime such that $p(\ell n+d) \equiv 0 \pmod{\ell}$ are $(5,4)$, $(7,5)$ and $(11,6)$.

1.1. **Statement of results.** In this paper we connect values of $p(\ell n + \delta_\ell)$ for $\ell \in \{5,7,11\}$ to central critical values of twists of modular L-functions. We begin by fixing some notation. Let $\ell \geq 5$ be prime. We define integers λ_ℓ and r_ℓ by

$$\lambda_\ell := \begin{cases} 9 & \ell = 5, \\ 8 & \ell = 7, \\ 14 & \ell = 11, \\ \frac{\ell-3}{2} & \ell \geq 13, \end{cases} \tag{1.4}$$

$$r_\ell := 24\left\lceil \frac{\ell}{24} \right\rceil - \ell\,. \tag{1.5}$$

Observe that r_ℓ is the least non-negative residue of $-\ell$ modulo 24. For non-negative integers n, define $D(\ell, n)$ by

$$D(\ell, n) := (-1)^{\lambda_\ell}(24n + r_\ell)\,. \tag{1.6}$$

Suppose that $D(\ell, n)$ is square-free. Then it is a fundamental discriminant (i.e., the discriminant of a quadratic number field). If t is a square-free integer, we denote by $\chi_t(\cdot)$ the Kronecker character associated to $\mathbb{Q}(\sqrt{t})$. In particular, we have $\chi_{D(\ell,n)}(\cdot) = \left(\frac{D(\ell,n)}{\cdot}\right)$. We define χ_{12} to be the character modulo 12 induced from the Kronecker character associated to $\mathbb{Q}(\sqrt{3})$.

For primes $\ell \geq 3$, let $(\frac{\cdot}{\ell})$ denote the Legendre symbol modulo ℓ. When $5 \leq \ell \leq 31$, one may verify that there exists a unique newform $G_\ell(z) = \sum_{m=1}^{\infty} a_\ell(m) q^m \in S_{2\lambda_\ell}(\Gamma_0(6))$ whose q-expansion is

$$G_\ell(z) := q + \left(\frac{2}{\ell}\right) 2^{\lambda_\ell - 1} q^2 + \left(\frac{3}{\ell}\right) 3^{\lambda_\ell - 1} q^3 + \dots\,. \tag{1.7}$$

Our result concerns the central critical values of the L-functions associated to $G_\ell(z)$ twisted by the Kronecker characters $\chi_{D(\ell,n)}$. For $s \in \mathbb{C}$ with $\Re(s) \geq \lambda_\ell + \frac{1}{2}$ and $D(\ell, n)$ square-free we define the associated L-function for $G_\ell(z)$ twisted by $\chi_{D(\ell,n)}$ in the usual way as

$$L(G_\ell \otimes \chi_{D(\ell,n)}, s) := \sum_{m=1}^{\infty} \chi_{D(\ell,n)}(m) a_\ell(m) m^{-s}\,.$$

This function has an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation centered at the line $s = \lambda_\ell$. We now state our main result.

Theorem 1.1. *Let $\ell \in \{5,7,11\}$, let $n \geq 0$ be an integer and let δ_ℓ, λ_ℓ, r_ℓ, $D(\ell,n)$, and $G_\ell(z)$ be defined as in* (1.2), (1.4), (1.5), (1.6), *and* (1.7). *Then if $D(\ell,n)$ is square-free, we have*

$$\frac{L(G_\ell \otimes \chi_{D(\ell,n)}, \lambda_\ell)(24n + r_\ell)^{\lambda_\ell - \frac{1}{2}}}{L(G_\ell \otimes \chi_{D(\ell,0)}, \lambda_\ell) r_\ell^{\lambda_\ell - \frac{1}{2}}} \equiv \frac{p(\ell n + \delta_\ell)^2}{\ell^2} \pmod{\ell}\,.$$

The method of proof entails finding a generating function for $\frac{p(\ell n+\delta_\ell)}{\ell}$ modulo ℓ that is a half-integral weight modular form to which we apply powerful theorems of Shimura and Waldspurger. The relevant modular forms lie in vector spaces of large dimension, which makes the necessary computations to confirm our results non-trivial.

Remark 1.2. Guo and Ono in [GO99] proved that Theorem 1.1 holds for primes $13 \leq \ell \leq 31$ (replacing $\frac{p(\ell n+\delta_\ell)^2}{\ell^2}$ with $\frac{p(\ell n+\delta_\ell)^2}{p(\delta_\ell)^2}$ in the statement). To prove our result, we extend their methods to account for the Ramanujan congruences.

Remark 1.3. In [GO99], Guo and Ono use their result for $13 \leq \ell \leq 31$ to show that the Bloch-Kato Conjecture implies that there is a close relationship between the ℓ-divisibility of $p(\ell n + \delta_\ell)$ and the ℓ-divisibility of orders of the Tate-Shafarevich groups of twists of motives associated to G_ℓ. They proved, assuming the Bloch-Kato conjecture, that ℓ divides the order of the relevant Tate-Shafarevich group if and only if $p(\ell n+\delta_\ell) \equiv 0 \pmod{\ell}$. For $\ell \in \{5, 7, 11\}$, an analogous argument assuming the Bloch-Kato Conjecture and using Theorem 1.1 would imply ℓ-divisibility results for orders of associated Tate-Shafarevich groups when $p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell^2}$.

Remark 1.4. One may verify results of this type for a fixed prime ℓ using our methods and those of Guo and Ono via a finite computation. Therefore, our ability to verify analogous results for fixed primes $\ell > 31$ only depends on our ability to do large computations. Moreover, since our method entails separate computations for each prime, we are unable to apply it to prove a uniform result for all primes $\ell \geq 5$.

For primes $\ell \in \{5, 7, 11\}$, we have $\frac{p(\ell n+\delta_\ell)^2}{\ell^2} \not\equiv 0 \pmod{\ell}$ if and only if $p(\ell n+\delta_\ell) \not\equiv 0 \pmod{\ell^2}$. Therefore we obtain the following result on the non-vanishing of central critical values of the L-functions of the type occurring in Theorem 1.1.

Corollary 1.5. *Let $n \geq 0$ be an integer for which $D(\ell, n)$ is square-free and suppose that $p(\ell n + \delta_\ell) \not\equiv 0 \pmod{\ell^2}$. Then we have*

$$L(G_\ell \otimes \chi_{D(\ell,n)}, \lambda_\ell) \neq 0\,.$$

To illustrate the corollary, we study the function

$$R_\ell(X) := \frac{\#\{0 \leq n \leq X : p(\ell n + \delta_\ell) \not\equiv 0 \pmod{\ell^2},\ D(\ell, n) \text{ square-free}\}}{\#\{0 \leq n \leq X : D(\ell, n) \text{ square-free}\}}\,. \tag{1.8}$$

This function gives the proportion of non-negative integers $n \leq X$ such that Corollary 1.5 implies the non-vanishing of $L(G_\ell \otimes \chi_{D(\ell,n)}, \lambda_\ell)$. Our calculations reveal the following data:

X	$R_5(X)$	$R_7(X)$	$R_{11}(X)$
1000	0.680829	0.364629	0.802186
10000	0.662284	0.371357	0.834448
100000	0.664233	0.371774	0.849149
900000	0.662061	0.372700	0.857340

The data is heavily influenced by extensions of (1.3) to modulus ℓ^2 (see for example [Kno93]) given by

$$\begin{aligned} p(25n+24) &\equiv 0 \pmod{25}\,, \\ p(49n+d) &\equiv 0 \pmod{49} \qquad \text{for } d \in \{19, 33, 40, 47\}\,, \\ p(121n+116) &\equiv 0 \pmod{121}]\,. \end{aligned} \tag{1.9}$$

For $\ell \in \{5, 7, 11\}$ we let $\mathcal{M}_\ell$ be the set of non-negative integers n such that $\ell n + \delta_\ell$ falls into the progressions in (1.9); i.e., we let

$$\mathcal{M}_\ell := \begin{cases} \{n \geq 0 : n \equiv 4 \pmod{5}\} & \ell = 5, \\ \{n \geq 0 : n \equiv 2, 4, 5, 6 \pmod{7}\} & \ell = 7, \\ \{n \geq 0 : n \equiv 10 \pmod{11}\} & \ell = 11\,. \end{cases}$$

Hence, we may renormalize $R_\ell(X)$ to account for the congruences (1.9) by defining

$$R'_\ell(X) := \frac{\#\{0 \leq n \leq X : p(\ell n + \delta_\ell) \not\equiv 0 \pmod{\ell^2},\ D(\ell, n) \text{ square-free},\ n \notin \mathcal{M}_\ell\}}{\#\{0 \leq n \leq X : D(\ell, n) \text{ square-free},\ n \notin \mathcal{M}_\ell\}}\,.$$

The data for $R'_\ell(X)$ is

X	$R'_5(X)$	$R'_7(X)$	$R'_{11}(X)$
1000	0.819135	0.847716	0.838517
10000	0.795368	0.851721	0.871350
100000	0.797205	0.850389	0.886614
900000	0.794490	0.852020	0.894960

Remark 1.6. We note that for primes $13 \leq \ell \leq 31$, the data in [GO99] show that $R_\ell(X)$ as in (1.8) seems to be slightly less than $1 - \frac{1}{\ell}$ as X gets large. Our data show that for primes $\ell \in \{5, 7, 11\}$, the normalization of $R_\ell(X)$ given by $R'_\ell(X)$ exhibits similar behavior as X gets large.

The paper is structured as follows. In Section 2 we introduce the necessary facts from the theory of modular forms that will be used in our proof. Section 3 contains the proof of our theorem. We include an appendix to discuss some of the details of our computations.

2. Facts on modular forms

In this section we give the necessary facts on modular forms that we require. For details on the theory, one may consult for example [Iwa97] or [Ono04].

Let $\mathfrak{h}$ be the complex upper half-plane, and let $q := e^{2\pi i z}$. For integers $k \geq 0$ and $N \geq 1$, and χ, a Dirichlet character modulo N, we let $M_k(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of weight k holomorphic modular forms on $\Gamma_0(N)$ with character χ. We denote by $S_k(\Gamma_0(N), \chi)$ the subspace of cusp forms. When χ is trivial, we often omit it.

For even $k \geq 4$, let B_k be the k-th Bernoulli number and define the Eisenstein series E_k by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n\,.$$

Then we have $E_k \in M_k(\Gamma_0(1))$, and for primes $p \geq 5$, we have $E_{p-1} \equiv 1 \pmod{p}$. We also require Dedekind's η-function, defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,,$$

and Ramanujan's Δ-function,

$$\Delta(z) := \eta(z)^{24} \in S_{12}(\Gamma_0(1))\,.$$

Next, we require certain standard operators on modular forms. For positive integers d, we define the operators U_d and V_d on formal power series in q by

$$\sum_{n=0}^{\infty} a(n)q^n|U_d := \sum_{n=0}^{\infty} a(dn)q^n,$$
$$\sum_{n=0}^{\infty} a(n)q^n|V_d := \sum_{n=0}^{\infty} a(n)q^{dn}.$$

When acting on spaces of modular forms, we have

$$V_d : M_k(\Gamma_0(N),\chi) \to M_k(\Gamma_0(Nd),\chi),$$

and if $d|N$, we have

$$U_d : M_k(\Gamma_0(N),\chi) \to M_k(\Gamma_0(N),\chi).$$

The U_d operator has the additional factorization property that

$$\left[\left(\sum_{n=0}^{\infty} a(n)q^{dn}\right)\left(\sum_{m=0}^{\infty} b(m)q^m\right)\right]|U_d = \left(\sum_{n=0}^{\infty} a(n)q^n\right)\left(\sum_{m=0}^{\infty} b(dm)q^m\right). \tag{2.1}$$

For positive integers m, the Hecke operator $T_{m,k,\chi}$ is an endomorphism on $M_k(\Gamma_0(N),\chi)$ and preserves cusp forms. For primes $p \nmid N$ we have

$$T_{p,k,\chi} := U_p + \chi(p)p^{k-1}V_p.$$

In the setting of modular forms with p-integral coefficients, it follows that $T_{p,k,\chi}$ and U_p agree modulo p.

We also recall the notion of twisting. Let ε be a Dirichlet character modulo M, and let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$. Then we define the twist of f by ε as

$$f(z) \otimes \varepsilon := \sum_{n=0}^{\infty} \varepsilon(n)a(n)q^n.$$

If $f(z) \in M_k(\Gamma_0(N),\chi)$, let N' be the conductor of χ, and define $\mathcal{N} := \mathrm{lcm}(N, N'M, M^2)$. Then we have $f(z) \otimes \varepsilon \in M_k(\Gamma_0(\mathcal{N}), \varepsilon^2\chi)$ (see for example [AL78]).

Next we require basic facts on half-integral weight modular forms. Let $4|N$, and let $\lambda \geq 0$ be an integer. We use $M_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N),\chi)$ to denote the space of holomorphic modular forms of weight $\lambda + \frac{1}{2}$ on $\tilde{\Gamma}_0(N)$ with Dirichlet character χ, and we denote by $S_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N),\chi)$ its subspace of cusp forms. For later purposes we note that $\eta(24z) \in S_{\frac{1}{2}}(\tilde{\Gamma}_0(576),\chi_{12})$. Let p be prime and let $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N),\chi)$. Then the half-integral weight Hecke operator $T_{p^2,\lambda,\chi}$ is defined by

$$f(z)|T_{p^2,\lambda+\frac{1}{2},\chi} := \sum_{n=0}^{\infty}\left(a(p^2n) + \chi^*(p)\left(\frac{n}{p}\right)a(n) + \chi^*(p^2)p^{2\lambda-1}a(n/p^2)\right)q^n,$$

where $\chi^*(n) := \left(\frac{-1}{n}\right)^{\lambda}\chi(n)$ and $a(n/p^2) := 0$ if $p^2 \nmid n$. As in the integer weight setting, $T_{p^2,\lambda+\frac{1}{2},\chi}$ is an endomorphism on $M_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N),\chi)$ and preserves cusp forms.

We now state the important theorems of Shimura and Waldspurger. Let N and χ be as above, and let λ and $t \geq 1$ be integers with t square-free. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N), \chi)$. We define

$$S_t(f)(z) := \sum_{n=1}^{\infty} A_{f,t}(n)q^n$$

by

$$\sum_{n=1}^{\infty} A_{f,t}(n)n^{-s} = L(\chi_t^{(\lambda)}, s-\lambda+1) \sum_{m=1}^{\infty} a_f(tm^2)m^{-s},$$

where $\chi_t^{(\lambda)}(\cdot) := \chi^*(\cdot)\chi_t(\cdot)$ is a character modulo Nt.

Theorem 2.1 (Shimura [Shi73])**.** *We have*

$$S_t(f) \in \begin{cases} M_{2\lambda}(\Gamma_0(N/2), \chi^2) & \text{if } \lambda = 1\,, \\ S_{2\lambda}(\Gamma_0(N/2), \chi^2) & \text{if } \lambda > 1\,. \end{cases}$$

Moreover, S_t commutes with Hecke operators:

$$S_t(f \mid T_{p^2,\lambda+\frac{1}{2},\chi}) = S_t(f) \mid T_{p,2\lambda,\chi^2}\,.$$

An immediate consequence is that if $f(z)$ is a half-integral weight eigenform, then $S_t(f)$ will be an integral weight eigenform. For our purposes we set $t = 1$ and define $S := S_1$.

We now state Waldspurger's result. Let $\lambda \geq 2$, let $f(z) \in S_{\lambda+\frac{1}{2}}(\tilde{\Gamma}_0(N), \chi)$ be an eigenform for almost all $T_{p^2,\lambda+\frac{1}{2},\chi}$, and let $F(z) = S(f)(z) \in S_{2\lambda}(\Gamma_0(N/2), \chi^2)$.

Theorem 2.2 (Waldspurger [Wal81])**.** *Suppose that n_1, n_2 are positive square-free integers such that for all $p|N$, we have $\frac{n_1}{n_2} \in \mathbb{Q}_p^{\times^2}$. Then we have*

$$a_f(n_1)^2 L(F \otimes (\chi^*)^{-1}\chi_{n_2}, \lambda)\chi(n_2)n_2^{\lambda-\frac{1}{2}} = a_f(n_2)^2 L(F \otimes (\chi^*)^{-1}\chi_{n_1}, \lambda)\chi(n_1)n_1^{\lambda-\frac{1}{2}}\,.$$

Remark 2.3. If $\lambda = 1$ and f is in the orthogonal complement of the space spanned by single variable theta series, then $S(f)$ is a cusp form, and Theorem 2.2 continues to hold.

3. Proof of main theorem

To prove Theorem 1.1 for $\ell \in \{5, 7, 11\}$, we will identify half-integer weight eigenforms f_ℓ that are generating functions for $\frac{p(\ell n+\delta_\ell)}{\ell}$. We will then show that $G_\ell \otimes \chi_{12} = S(f_\ell)$ (with G_ℓ as in (1.7)); to conclude we will apply Waldspurger's Theorem.

Lemma 3.1. *For $\ell \in \{5, 7, 11\}$, define*

$$f_\ell(z) := \begin{cases} \eta(24z)^{19} & \ell = 5\,, \\ \eta(24z)^{17} & \ell = 7\,, \\ \eta(24z)^{13}E_8(24z) & \ell = 11\,. \end{cases}$$

Then we have

$$\sum_{n\geq 0} \frac{p(\ell n+\delta_\ell)}{\ell} q^{24n+r_\ell} \equiv f_\ell \pmod{\ell}\,.$$

Proof. To prove the congruence we examine $\Delta(z)^{(\ell^2-1)/24} \mid U_\ell$. We first note that the forms $\Delta(z)^{(\ell^2-1)/24} \mid T_{\ell,\frac{\ell^2-1}{2}} \in S_{\frac{\ell^2-1}{2}}(\Gamma_0(1))$ have integer coefficients. For $\ell \in \{5,7,11\}$, one may easily verify that the coefficients of $\Delta(z)^{(\ell^2-1)/24} \mid T_{\ell,\frac{\ell^2-1}{2}}$ of index up to $(\ell^2-1)/24$ are congruent to zero modulo ℓ. By a theorem of Sturm (see [Ono04], Thm. 2.58), it follows that

$$\Delta(z)^{(\ell^2-1)/24} \mid T_\ell \equiv 0 \pmod{\ell}.$$

Hence, the forms $\frac{1}{\ell}\Delta(z)^{(\ell^2-1)/24} \mid T_{\ell,\frac{\ell^2-1}{2}} \in S_{(\ell^2-1)/2}(\Gamma_0(1))$ have integer coefficients. Applying Sturm's Theorem again together with the congruence $E_{\ell-1} \equiv 1 \pmod{\ell}$, we find that

$$\frac{1}{\ell}\Delta(z)^{(\ell^2-1)/24} \mid T_{\ell,\frac{\ell^2-1}{2}} \equiv \begin{cases} \Delta(z) & \pmod{\ell} \quad \ell = 5, \\ \Delta(z)E_6(z)^2 \equiv \Delta(z) & \pmod{\ell} \quad \ell = 7, \\ \Delta(z)E_8(z)E_{10}(z)^4 \equiv \Delta(z)E_8(z) & \pmod{\ell} \quad \ell = 11. \end{cases}$$

Since U_ℓ and $T_{\ell,\frac{\ell^2-1}{2}}$ agree on modular forms with integer coefficients modulo ℓ, we have

$$\frac{1}{\ell}\Delta(z)^{(\ell^2-1)/24}|U_\ell \equiv \begin{cases} \Delta(z) & \pmod{\ell} \quad \ell = 5,7, \\ \Delta(z)E_8(z) & \pmod{\ell} \quad \ell = 11. \end{cases} \tag{3.1}$$

Using (1.1), (1.2), and (2.1) we see, on the other hand, that

$$\begin{aligned} \frac{1}{\ell}\Delta(z)^{(\ell^2-1)/24} \mid U_\ell &= \frac{1}{\ell}q^{(\ell^2-1)/24}\prod_{n=1}^{\infty}\frac{(1-q^n)^{\ell^2}}{(1-q^n)} \mid U_\ell \\ &\equiv \prod_{n=1}^{\infty}(1-q^n)^\ell\left(\sum_{n=(\ell^2-1)/24}^{\infty}\frac{p(n-\frac{\ell^2-1}{24})}{\ell}q^n\right) \mid U_\ell \\ &\equiv \prod_{n=1}^{\infty}(1-q^n)^\ell \sum_{n=\lfloor\frac{\ell}{24}\rfloor+1}^{\infty}\frac{p(\ell n-\frac{\ell^2-1}{24})}{\ell}q^n \\ &\equiv q^{\lfloor\frac{\ell}{24}\rfloor+1}\prod_{n=1}^{\infty}(1-q^n)^\ell\sum_{n=0}^{\infty}\frac{p(\ell n+\delta_\ell)}{\ell}q^n \pmod{\ell}. \end{aligned} \tag{3.2}$$

By comparing (3.1) and (3.2) and solving for $\sum_{n=0}^{\infty}\frac{p(\ell n+\delta_\ell)}{\ell}q^n$, we see that

$$\sum_{n=0}^{\infty}\frac{p(\ell n+\delta_\ell)}{\ell}q^n \equiv \begin{cases} \prod_{n=1}^{\infty}(1-q^n)^{19} & \pmod{\ell} \quad \ell = 5, \\ \prod_{n=1}^{\infty}(1-q^n)^{17} & \pmod{\ell} \quad \ell = 7, \\ \prod_{n=1}^{\infty}(1-q^n)^{13}E_8(z) & \pmod{\ell} \quad \ell = 11. \end{cases}$$

By substituting q^{24} for q and multiplying both sides by q^{r_ℓ}, in each case we get our result. □

Next, we observe that the forms f_ℓ are eigenforms for the Hecke operators.

Lemma 3.2 (Garvan [Gar07])**.** *If* $\ell \in \{5, 7, 11\}$*, then* $f_\ell \in S_{\lambda_\ell+\frac{1}{2}}(\tilde{\Gamma}_0(576), \chi_{12})$ *is a Hecke eigenform for all Hecke operators* $T_{p^2,\lambda_\ell+\frac{1}{2},\chi_{12}}$*.*

This is proved as part of Corollary 3.2 in [Gar07]. Since $G_\ell \in S_{2\lambda_\ell}(\Gamma_0(6))$ is a newform, it follows by Theorem 3.1 (and its corollary) of [AL78] that $G_\ell \otimes \chi_{12} \in S_{2\lambda_\ell}(\Gamma_0(144))$ is a newform. Moreover, we prove the following:

Lemma 3.3. *If* $\ell \in \{5, 7, 11\}$*, then we have* $G_\ell \otimes \chi_{12} = S(f_\ell)$*.*

Proof. For $\ell \in \{5, 7, 11\}$, we have $S(f_\ell)$ and $G_\ell \otimes \chi_{12} \in S_{2\lambda_\ell}(\Gamma_0(288))$, and we define $M_\ell := \dim S_{2\lambda_\ell}(\Gamma_0(288))$. Applying standard formulas (see for example [Ono04]), we find that

$$M_\ell = \begin{cases} 800 & \ell = 5\,, \\ 704 & \ell = 7\,, \\ 1280 & \ell = 11\,. \end{cases}$$

Therefore, to prove the lemma, we calculate the first M_ℓ coefficients of $S(f_\ell)$ and $G_\ell \otimes \chi_{12}$ and verify that they agree. For details of this calculation, see Section 4. $\square$

We finalize the proof of Theorem 1.1 by applying Theorem 2.2 with $f = f_\ell$, $F = G_\ell \otimes \chi_{12}$, $n_1 = r_\ell$ and $n_2 = 24n + r_\ell$. Since $n_1 \equiv n_2 \pmod{24}$, we have $\frac{n_2}{n_1} \in \mathbb{Q}_p^{\times 2}$ for $p \in \{2, 3\}$. We calculate that $\chi_{12}^*(\cdot)(\frac{24n+r_\ell}{\cdot}) = (\frac{(-1)^{\lambda_\ell}(24n+r_\ell)}{\cdot}) = \chi_{D(\ell,n)}(\cdot)$. Thus, we have

$$\frac{L(G_\ell \otimes \chi_{D(\ell,n)}, \lambda_\ell)(24n + r_\ell)^{\lambda_\ell - \frac{1}{2}}}{L(G_\ell \otimes \chi_{D(\ell,0)}, \lambda_\ell)(r_\ell)^{\lambda_\ell - \frac{1}{2}}} = \frac{a_{f_\ell}(24n + r_\ell)^2}{a_{f_\ell}(r_\ell)^2}\,.$$

Noting that $a_{f_\ell}(24n+r_\ell) \equiv \frac{p(\ell n+\delta_\ell)}{\ell}$ modulo ℓ and $a_{f_\ell}(r_\ell) = 1$, the proof is complete.

4. Appendix

To compare $S(f_\ell)$ and $G_\ell \otimes \chi_{12}$, we will need to compute the first M_ℓ coefficients for each form. Set p_ℓ to be the largest prime less than M_ℓ. Then to compute $S(f_\ell)$ we must compute the first $p_\ell^2 r_\ell$ coefficients of f_ℓ. To compute these coefficients we employ well-known formulas for $\eta(24z)$ and $\eta(24z)^3$:

$$\eta(24z) = 1 + \sum_{n \geq 1} (-1)^n (q^{12n(3n-1)} + q^{12n(3n+1)}),$$

$$\eta(24z)^3 = \sum_{n \geq 0} (-1)^n (2n + 1) q^{3(2n+1)^2}\,.$$

The first formula is Euler's Pentagonal Number Theorem, and the second follows from Jacobi's Triple Product Identity. These formulas allow us to rapidly calculate the necessary number of coefficients for $\eta(24z)$ and $\eta(24z)^3$ on a computer, which we use to build $\eta^{13}(24z)$, $\eta^{17}(24z)$, and $\eta^{19}(24z)$.

For the most efficient use of computer memory available to us, the powers of η were encoded as arrays and then saved as text files. To multiply them together, we wrote simple programs which kept our memory use under 2 GB of RAM. The text file for the coefficients of $\eta^{19}(24z)$ was approximately 82 MB.

The calculation of the necessary coefficients for $S(f_5)$ and $S(f_7)$ is straightforward from the definition. However, for $f_{11} = \eta^{13}(24z)E_8(24z)$, the Eisenstein factor makes the size of the coefficients significantly larger. Therefore, to conserve file space, we employ a Chinese Remainder Theorem argument.

For the integer weight newforms G_ℓ, we follow a similar argument as in [GO99]. We identify $G_\ell^* \in S_{2\lambda_\ell}(\Gamma_0(6))$, which are linear combinations of η-products, such that $G_\ell \otimes \chi_1 = G_\ell^* \otimes \chi_1$ in $S_{2\lambda_\ell}(\Gamma_0(36))$, where χ_1 denotes the trivial character modulo 6. Hence, we have $G_\ell \otimes \chi_{12} = G_\ell^* \otimes \chi_{12}$. From the bound for the dimension of $S_{2\lambda_\ell}(\Gamma_0(36))$ coming from the valence formula, we prove this equality of modular forms by checking that the first $12\lambda_\ell$ coefficients of the forms agree. One uses standard techniques to compute the required number of coefficients of the newform G_ℓ, and the forms G_ℓ^* are given explicitly as follows:

$$\begin{aligned} G_5^* :=& - 1140\eta(z)^{30}\eta(3z)^6 - 40643208\eta(z)^6\eta(3z)^6\eta(6z)^{24} \\ &- 3147616\eta(z)\eta(2z)^{25}\eta(3z)^5\eta(6z)^5 + \eta(z)^{29}\eta(2z)^5\eta(3z)\eta(6z) \mid U_2 \\ &+ 36120492\eta(2z)^6\eta(3z)^{24}\eta(6z)^6 - 42133284\eta(z)^{10}\eta(2z)^4\eta(3z)^2\eta(6z)^{20} \\ &+ 2184396\eta(z)^{10}\eta(2z)^{16}\eta(3z)^2\eta(6z)^8 - 71788\eta(z)^{25}\eta(2z)\eta(3z)^5\eta(6z)^5\,, \end{aligned}$$

$$\begin{aligned} G_7^* :=& - 4110102\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^{26} - 1239300\eta(z)^2\eta(2z)^{14}\eta(3z)^2\eta(6z)^{14} \\ &+ 1166400\eta(z)^3\eta(2z)^9\eta(3z)^7\eta(6z)^{13} - 729\eta(z)^2\eta(2z)^2\eta(3z)^{26}\eta(6z)^2 \\ &- 13414\eta(z)^2\eta(2z)^{26}\eta(3z)^2\eta(6z)^2 + 87\eta(z)^{26}\eta(2z)^2\eta(3z)^2\eta(6z)^2 \\ &+ \eta(z)^{26}\eta(2z)^2\eta(3z)^2\eta(6z)^2 \mid U_2\,, \end{aligned}$$

$$\begin{aligned} G_{11}^* :=&\eta(z)^{50}\eta(2z)^2\eta(3z)^2\eta(6z)^2 \mid U_3 \\ &+ \frac{6104578068776124145}{751160222919}\eta(z)^{50}\eta(2z)^2\eta(3z)^2\eta(6z)^2 \\ &+ \frac{13103131110144431}{8012375711136}\eta(z)^{45}\eta(2z)^3\eta(3z)\eta(6z)^7 \mid U_2 \\ &- \frac{3299973272427100336}{751160222919}\eta(z)^{40}\eta(2z)^{16} \\ &+ \frac{309477427592855}{27820748997}\eta(z)^{38}\eta(2z)^2\eta(3z)^{14}\eta(6z)^2 \mid U_2 \\ &+ \frac{17128468858079251316}{751160222919}\eta(z)^{37}\eta(2z)^7\eta(3z)^9\eta(6z)^3 \\ &- \frac{162517745107473935}{8012375711136}\eta(z)^{37}\eta(2z)^7\eta(3z)^9\eta(6z)^3 \mid U_2 \\ &- \frac{25508747927970736722 88}{83462246991}\eta(z)^{36}\eta(3z)^4\eta(6z)^{16} \\ &- \frac{56155589295498479006 8}{83462246991}\eta(z)^{33}\eta(2z)^3\eta(3z)^{13}\eta(6z)^7 \\ &+ \frac{354025648394158659737 6}{751160222919}\eta(z)^{31}\eta(2z)^{13}\eta(3z)^3\eta(6z)^9 \\ &- \frac{8342412746725559808}{38162893}\eta(z)^{31}\eta(2z)\eta(3z)^3\eta(6z)^{21} \\ &+ \frac{201341803552556544}{38162893}\eta(z)^{24}\eta(3z)^{16}\eta(6z)^{16} \\ &+ \frac{2388734537489757470 72}{751160222919}\eta(z)^{16}\eta(2z)^{40}\,. \end{aligned}$$

Finally we verify that $G_\ell^* \otimes \chi_{12} = S(f_\ell)$. For $\ell = 11$ we need bounds on the size of the coefficients in order to employ the Chinese Remainder Theorem. Based on

Deligne's proof of the Weil Conjectures, we have the following bound for the size of the coefficients of newforms.

If $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N))$ is a newform, then

$$|a_f(n)| \leq \sum_{1 \leq d | n} n^{(k-1)/2}.$$

This implies that $|a_{S(f_{11})}(n)|, |a_{G_{11}}(n)| < 10^{44}$ for $n \leq 1280$. Finally, we finish the proof of Lemma 3 by verifying that the first 1,280 coefficients of $G_{11}^{*} \otimes \chi_{12}$ and $S(f_{11})$ agree modulo p, where p ranges over the first 8 primes greater than 10^6.

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