ARITHMETIC OF THE 13-REGULAR PARTITION FUNCTION MODULO 3

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Abstract. Let $b_{13}(n)$ denote the number of 13-regular partitions of n. We study in this paper the behavior of $b_{13}(n)$ modulo 3 where $n \equiv 1 \pmod 3$. In particular, we identify an infinite family of arithmetic progressions modulo arbitrary powers of 3 such that $b_{13}(n) \equiv 0 \pmod 3$.

1. Introduction

Let $p(n)$ be the number of partitions of a positive integer n: the number of non-increasing sequences of positive integers whose sum is n. By convention, we agree that $p(0) := 1$. Partitions play an important role in number theory, combinatorics, representation theory, and mathematical physics. Euler showed that the generating function for $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} (1-q^m)^{-1}.$$

Among the most famous results on $p(n)$ are the Ramanujan congruences. For all non-negative integers n, these are

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Recent works of Ahlgren and Ono, [9], [1], and [3], show that for all moduli m coprime to 6, there are infinitely many arithmetic progressions $An+B$ such that

$$p(An+B) \equiv 0 \pmod m.$$

These works motivate an investigation of congruence phenomena for other types of partition functions.

Let k be a positive integer. In this paper, we study congruences for k-regular partitions, which we now define. We say that a partition of n is k-regular if none of its parts are divisible by k. We let $b_k(n)$ denote the number of k-regular partitions of n, and we agree that $b_k(0) := 1$. The generating function for $b_k(n)$ is

$$\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{km})}{(1-q^m)}. \tag{1.1}$$

We now briefly describe some of the known facts on divisibility and distribution of $b_k(n) \pmod m$. Gordon and Ono in [7] prove the following result. Let k be a positive integer and

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let p be a prime divisor of k. If $p^{\mathrm{ord}_p k} \geq \sqrt{k}$, then for every positive integer j, the arithmetic density of integers n for which $b_k(n) \equiv 0 \pmod{p^j}$ is 1. In [12] and [13], Penniston proves results on the distribution of $b_k(n)$ in residue classes modulo a prime p. Specifically, in [13], if $k \neq p$ are primes with $3 \leq k \leq 23$ and $p \geq 5$, then the density of positive integers n such that $b_k(n) \equiv 0 \pmod{p}$ is at least $\frac{p+1}{2p}$ if $p \nmid k-1$ and at least $\frac{p-1}{p}$ if $p \mid k-1$. Treener in [14] extends this result to all primes $k \geq 3$ not equal to p. In [8] and [6], the authors use the theory of complex multiplication to precisely describe those n for which $b_\ell(n) \equiv 0 \pmod{\ell}$ when $\ell \in \{3, 5, 7, 11\}$. We also recall the classical result of Euler which states that $b_2(n)$, the number of partitions into odd parts, agrees with the number of partitions of n into distinct parts, often denoted by $Q(n)$. Recent works on the arithmetic of $Q(n)$ include, for example, [2] and [11].

In [5], the authors prove that there are infinitely many Ramanujan-type congruences modulo 2 for $b_5(n)$ and $b_{13}(n)$ in arithmetic progressions of even numbers. Additionally, they prove that for every $n \geq 0$,

$$\begin{aligned} b_5(20n+5) &\equiv 0 \pmod{2} \\ \text{and } b_5(20n+13) &\equiv 0 \pmod{2}. \end{aligned}$$

Further, the authors in [5] make the following conjecture about $b_{13}(n)$.

Conjecture 1 ([5]). *For all integers n and ℓ with $n \geq 0$ and $\ell \geq 2$, we have*

$$b_{13}\left(3^\ell n + \frac{5 \cdot 3^{\ell-1} - 1}{2}\right) \equiv 0 \pmod{3}.$$

The authors verified their conjecture for $2 \leq \ell \leq 6$.

In this paper we further investigate the behavior of $b_{13}(n)$ modulo 3 for n in certain arithmetic progressions modulo powers of 3. We prove the following congruence relating values of $b_{13}(n)$ for $n \equiv 1 \pmod{3}$.

Theorem 1.1. *For all integers $n \geq 0$ we have*

$$b_{13}(3n+1) \equiv -b_{13}(9n+4) \pmod{3}.$$

Since $9n+4$ is a subprogression of $3n+1$, we observe that an equivalent statement to Theorem 1.1 is that for integers n and ℓ with $n \geq 0$ and $\ell \geq 1$, we have

$$b_{13}(3n+1) \equiv (-1)^{\ell+1} b_{13}\left(3^\ell n + \frac{3^\ell - 1}{2}\right) \pmod{3}. \tag{1.2}$$

Using Theorem 1.1 we prove

Theorem 1.2. *Conjecture 1 is true.*

To prove Theorems 1.1 and 1.2, we relate the generating function for $b_{13}(3n+1)$ to a modular form and study its image under certain operators. In Section 2, we give the necessary background on modular forms. In Section 3, we prove the theorems.

2. Background on Modular Forms

Let $\mathfrak{h}$ be the complex upper-half plane, and let $q := e^{2\pi i z}$. For integers $k \geq 0$ and $N \geq 1$, and for χ, a Dirichlet character modulo N, we let $M_k(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space

of weight k holomorphic modular forms on $\Gamma_0(N)$ with character χ. We require Dedekind's eta-function, defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \,.$$

It serves as an important building block for modular forms, as we now describe. Let N be a positive integer. We say that a function $f(z)$ is an eta-quotient if it has the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta} \,,$$

where each r_δ is an integer. One may apply the following theorem of Gordon, Hughes, and Newman to determine whether an eta-quotient transforms properly with respect to the congruence subgroup $\Gamma_0(N)$.

Theorem 2.1 (Gordon, Hughes, and Newman, [10]). *If $f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}$ with the additional properties that*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$N \sum_{\delta | N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24} \,,$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s := \prod_{\delta | N} \delta^{r_\delta}$.

We note that $\eta(z)$ is holomorphic and non-zero on $\mathfrak{h}$. Therefore, the poles of an eta-quotient satisfying Theorem 2.1, if it has any, must be supported at cusps. One may use the following theorem of Ligozat to compute the orders of vanishing at cusps for eta-quotients satisfying Theorem 2.1. If all such orders are non-negative then the eta-quotient is a holomorphic modular form.

Theorem 2.2 (Ligozat, [10]). *Let c, d, and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.1 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta | N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta} \,.$$

For example, Theorems 2.1 and 2.2 imply that

$$\frac{\eta(13z)^{37}}{\eta(z)} \in M_{18}(\Gamma_0(13), \chi_{13}) \,, \tag{2.1}$$

where χ_{13} is the Kronecker character associated to $\mathbb{Q}(\sqrt{13})$. We require this modular form for the proofs of Theorems 1.1 and 1.2. Next, we require certain standard operators on modular forms. For positive integers d, we define the operator U_d on formal power series in q by

$$\sum_{n=0}^{\infty} a(n)q^n | U_d := \sum_{n=0}^{\infty} a(dn)q^n \,.$$

When acting on spaces of modular forms, if $d|N$, we have

$$U_d : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi) \,.$$

The U_d operator has the additional factorization property that

$$\left[\left(\sum_{n=0}^{\infty} a(n)q^{dn} \right) \left(\sum_{m=0}^{\infty} b(m)q^m \right) \right] | U_d = \left(\sum_{n=0}^{\infty} a(n)q^n \right) \left(\sum_{m=0}^{\infty} b(dm)q^m \right) \,. \tag{2.2}$$

For positive integers m, the Hecke operator T_m is an endomorphism on $M_k(\Gamma_0(N), \chi)$. Let p be prime. Then for $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ we have

$$f \mid T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{k-1}a(n/p) \right) q^n \,,$$

where if $p \nmid n$ we agree that $a(n/p) = 0$. In the setting of modular forms of weight greater than 1 with p-integral coefficients, it follows that T_p and U_p agree modulo p.

We also recall the notion of twisting. Let ε be a Dirichlet character modulo M, and let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$. Then we define the twist of f by ε as

$$f(z) \otimes \varepsilon := \sum_{n=0}^{\infty} \varepsilon(n)a(n)q^n \,.$$

If $f(z) \in M_k(\Gamma_0(N), \chi)$, let N' be the conductor of χ, and define $\mathcal{N} := \mathrm{lcm}(N, N'M, M^2)$. Then we have $f(z) \otimes \varepsilon \in M_k(\Gamma_0(\mathcal{N}), \varepsilon^2\chi)$ (see for example [4]).

We now state an important theorem of Sturm which provides a method to test whether two modular forms with integer coefficients are congruent modulo a prime. Let m be a positive integer, and let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ be a modular form with integer coefficients. Then we define $\mathrm{ord}_m(f(z))$ to be the smallest n such that $a(n) \not\equiv 0 \pmod{m}$; if no such n exists, then we say that $\mathrm{ord}_m(f(z)) := \infty$.

Theorem 2.3 (Sturm, [10])**.** *Suppose that N is a positive integer, p is prime and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If*

$$\mathrm{ord}_p(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)] \,,$$

then we have $f(z) \equiv g(z) \pmod{p}$.

We note that

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\ell \text{ prime}: \ell | N} \left(1 + \frac{1}{\ell} \right) \,.$$

3. Proof of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we first find an integer weight modular form which is congruent to the generating function for $b_{13}(3n+1)$ modulo 3. Next we will analyze the behavior of this form when it is hit by the T_3 operator and when it is twisted by characters modulo 3. Theorems 1.1 and 1.2 arise when we relate this behavior back to the values of the 13-regular partition function.

We begin by realizing the generating function for $b_{13}(3n+1)$ as a modular form. For all $m \in \mathbb{Z}$, define $G_m(z)$ by

$$G_m(z) := \eta(6z)^{2m-1}\eta(78z)^{25-2m} = q^{81-6m} + \dots . \tag{3.1}$$

Using Theorems 2.1 and 2.2, we see for $0 \le m \le 13$ that $G_m(z) \in M_{12}(\Gamma_0(312), \chi_{13})$, where as before χ_{13} is the Kronecker character attached to $\mathbb{Q}(\sqrt{13})$. We define

$$\begin{aligned} H(z) :=& 2G_0(z) + 2G_2(z) + G_3(z) + 2G_4(z) + G_5(z) + 2G_6(z) \\ &+ G_7(z) + 2G_8(z) + G_9(z) + 2G_{10}(z) + G_{11}(z) + G_{13}(z) \\ =& q^3 - 25q^9 + 276q^{15} + \dots \in M_{12}(\Gamma_0(312), \chi_{13}) . \end{aligned} \tag{3.2}$$

Proposition 3.1. *We have*

$$H(z) \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^{6n+3} \pmod 3 .$$

Proof of Proposition 3.1. Using (1.1) and (2.1), we start with

$$\frac{\eta(13z)^{37}}{\eta(z)} = q^{20}\prod_{n\ge 1}(1-q^{13n})^{36}\sum_{n=0}^{\infty} b_{13}(n)q^n \ \in M_{18}(\Gamma_0(13), \chi_{13}) . \tag{3.3}$$

The first step is to explicitly identify a form $h(z) \in M_{18}(\Gamma_0(13), \chi_{13})$ such that

$$h(z) \equiv \frac{\eta(13z)^{37}}{\eta(z)} \mid T_3 \pmod 3 . \tag{3.4}$$

For all $m \in \mathbb{Z}$, define $g_m(z)$ by

$$g_m(z) := \eta(z)^{2m-1}\eta(13z)^{37-2m} = q^{20-m} + \dots . \tag{3.5}$$

Then by Theorems 2.1 and 2.2, we have $g_m(z) \in M_{18}(\Gamma_0(13), \chi_{13})$ for $-1 \le m \le 20$. We define $h(z)$ as

$$\begin{aligned} h(z) :=& 2g_0(z) + 2g_2(z) + g_3(z) + 2g_4(z) + g_5(z) + 2g_6(z) \\ &+ g_7(z) + 2g_8(z) + g_9(z) + 2g_{10}(z) + g_{11}(z) + g_{13}(z) \\ =& q^7 - 25q^8 + 276q^9 + \dots \in M_{18}(\Gamma_0(13), \chi_{13}) . \end{aligned} \tag{3.6}$$

Using Sturm's Theorem, we see that this $h(z)$ satisfies (3.4).

On the other hand, using (2.2), (3.3), and the fact that the T_3 and U_3 operators agree modulo 3 on modular forms with integer coefficients and weights greater than 1, we find that

$$\begin{aligned}\frac{\eta(13z)^{37}}{\eta(z)} \mid T_3 &\equiv \prod_{n\geq 1}(1-q^{13n})^{12}\left[\sum_{n=0}^{\infty} b_{13}(n)q^{n+20}\right] \mid T_3 \\ &\equiv \prod_{n\geq 1}(1-q^{13n})^{12}\sum_{n=0}^{\infty} b_{13}(3n+1)q^{n+7} \pmod 3\,. \end{aligned} \tag{3.7}$$

Hence, combining (3.4) and (3.7), we see that

$$\frac{h(z)}{q^7\prod_{n\geq 1}(1-q^{13n})^{12}} \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^n \pmod 3\,. \tag{3.8}$$

We also note from (3.1) and (3.5) that

$$G_m(z) = \frac{g_m(6z)}{q^{39}\prod_{n\geq 1}(1-q^{78n})^{12}}\,. \tag{3.9}$$

Thus, combining (3.2), (3.6), (3.8) and (3.9), we see that

$$H(z) \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^{6n+3} \pmod 3\,.$$

□

The next proposition gives the image modulo 3 of $H \mid T_3$ under T_3 and twists by characters modulo 3.

Proposition 3.2. *We have*

$$H(z) \mid T_3 \equiv -H(z) \mid T_3^2 \pmod 3\,, \tag{3.10}$$

and

$$(H(z) \mid T_3) \otimes \chi^{triv} \equiv (H(z) \mid T_3) \otimes \chi_{-3} \pmod 3\,, \tag{3.11}$$

where χ^{triv} and χ_{-3} are respectively the trivial and nontrivial characters modulo 3.

Proof of Proposition 3.2. The validity of the statements follows from calculating enough coefficients of each form and then applying Sturm's Theorem. The forms in (3.10) lie in $M_{12}(\Gamma_0(312), \chi_{13})$ whose Sturm bound is 672. Therefore, it suffices to verify that $\mathrm{ord}_3(H(z) \mid T_3 + H(z) \mid T_3^2) > 672$. For (3.11), we see that $(H(z) \mid T_3) \otimes \chi^{triv}$ and $(H(z) \mid T_3) \otimes \chi_3$ lie in $M_{12}(\Gamma_0(936), \chi_{13})$, whose Sturm bound is 2016. Thus we verify that $\mathrm{ord}_3((H(z) \mid T_3) \otimes \chi^{triv} - (H(z) \mid T_3) \otimes \chi_3) > 2016$. □

By combining Propositions 3.1 and 3.2, we now prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. An immediate consequence of Proposition 3.1 is that

$$H(z) \mid T_3 \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^{2n+1} \pmod 3\,. \tag{3.12}$$

Theorem 1.1 follows immediately from applying (3.10) of Proposition 3.2 to (3.12).

For Theorem 1.2, (3.11) of Proposition 3.2 asserts that the nth coefficient of $H \mid T_3$ is congruent to zero modulo 3 whenever $n \equiv 2 \pmod 3$. Hence, using (3.12), for all non-negative integers n we have

$$b_{13}(3(3n+2)+1) = b_{13}(9n+7) \equiv 0 \pmod 3\,. \tag{3.13}$$

By (1.2) we have

$$\begin{aligned} b_{13}(9n+7) = b_{13}(3(3n+2)+1) &\equiv (-1)^{\ell+1} b_{13}\left(3^{\ell}(3n+2) + \frac{3^{\ell}-1}{2}\right) \pmod 3 \\ &\equiv (-1)^{\ell+1} b_{13}\left(3^{\ell+1}n + \frac{5\cdot 3^{\ell}-1}{2}\right). \end{aligned} \tag{3.14}$$

Finally, we combine (3.13) and (3.14) to get our result. □

Remark. In [5], the authors proved (3.13) by showing that

$$\frac{\eta(13z)^{37}}{\eta(z)} \mid T_3^2 \equiv 0 \pmod 3\,.$$

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