Neither Chocolate Candy Nor Rap Music: The Real M&m's

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Begin with any three real numbers... Say 25, -7, and 29. Note that their **median** is 25. We will construct a sequence (x_n), where $x_1 = 25$, $x_2 = -7$, and $x_3 = 29$. The fourth term, x_4, is chosen so that the **mean** of x_1, x_2, x_3, x_4 is 25 (the **median** of x_1, x_2, x_3). The number that satisfies this condition is $x_4 = 53$. Now, we repeat the procedure to obtain x_5. The **median** of 25, -7, 29, 53 is 27.[1] Choose x_5 so that the **mean** of x_1, x_2, x_3, x_4, x_5 is 27. The number satisfying this condition is $x_5 = 35$. If we continue to construct the sequence in this manner, then we get the following terms:

$$25, -7, 29, 53, 35, 39, 50, 56, 53, 57, 99.5, 110.5, 69.5, 72.5, 53, 53, 53, ...$$

It becomes constant at 53. Consider a different set of three numbers: -50, -5, 5. If we repeat the above procedure, then we obtain the sequence

$$-50, -5, 5, 30, 20, 30, 57.5, 72.5, 65, 75, 30, 30, 30, ...$$

Again, a constant sequence at the value of 30. Upon further investigation, it seems that every such sequence becomes eventually constant. This begs the question, "Does this always happen?" This question will be discussed at length in this paper.

So, in general, the sequence construction begins with three values, x_1, x_2, x_3. For $k \geq 3$, calculate the **median** of the first k values of the sequence, call it m_k. Define x_{k+1} to be the unique real number such that the **mean** of the first $k + 1$ values of the sequence equals m_k. Such a sequence will be referred to as an *M&m sequence* for the remainder of the paper. If σ is any sequence of three real numbers, then we will denote the corresponding M&m sequence by $\mu(\sigma)$. We will denote the **median** of the first n terms of $\mu(\sigma)$ by $m_n(\sigma)$ [as mentioned above] and the **mean** of the first n terms of $\mu(\sigma)$ by $M_n(\sigma)$. So, if we have $\mu(\sigma) = (x_n : n \in \omega)$, then the value x_{n+1} is the unique number satisfying $M_{n+1}(\sigma) = m_n(\sigma)$. If no confusion can arise, we will use M_n and m_n instead of $M_n(\sigma)$ and $m_n(\sigma)$.

[1] Note that we are using one of several possible definitions for the median of an even number of arguments. In this paper, we will take the average of the two middle numbers.

An M&m sequence $\mu(\sigma) = (x_n : n \in \omega)$ is *convergent* if $\lim_{n\to\infty} x_n$ exists. We say that $\mu(\sigma)$ is *stable* if the sequence is eventually constant. In other words, $\mu(\sigma)$ is stable if and only if there exists some k for which $x_n = x_k$ for each $n \geq k$. In this case, the smallest such k is called the *length* of the sequence and x_k is called its *stable value.* Returning to our first example, the M&m sequence

$$25, -7, 29, 53, 35, 39, 50, 56, 53, 57, 99.5, 110.5, 69.5, 72.5, 53, 53, 53, ...$$

has length 15 and a stable value of 53. This brings us to the following conjecture:

The M&m Conjecture *Every M&m sequence is stable.*

Prior to proceeding, there are several simplifications that will be made. Suppose that we have an M&m sequence $\mu(\sigma) = (x_n : n \in \omega)$. First, since the order of x_1, x_2, x_3 does not affect $\mu(\sigma)$, we suppose that $x_1 \leq x_2 \leq x_3$. If $x_1 = x_2 = x_3$, then clearly $\mu(\sigma)$ is a constant sequence and thus, not terribly interesting. If any two of x_1, x_2, x_3 are equal, then $\mu(\sigma)$ has length 5, again, quite uninteresting. To see this, suppose that $x_1 = x_2 < x_3$. Well, we know that $m_3 = x_2 = x_1$ and x_4 will be chosen so that $\dfrac{x_1 + x_2 + x_3 + x_4}{4} = m_3$. By substitution, we get that $x_1 + x_1 + x_3 + x_4 = 4x_1$, and so $x_3 + x_4 = 2x_1$. Since $x_3 > x_1$ by assumption, we have $x_4 < x_1$. It follows that $m_4 = \dfrac{x_1 + x_2}{2} = \dfrac{x_1 + x_1}{2} = x_1$. Since x_5 is chosen so that $\dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = m_4$, we get $x_1 + x_1 + x_3 + x_4 + x_5 = 5x_1$. Since $x_3 + x_4 = 2x_1$, we can substitute to get $x_5 = x_1$. Thus, $m_5 = x_1$ and it is clear that $x_i = x_1$ for $i \geq 5$. Thus, $\mu(\sigma)$ has length 5. The proof is similar for $x_1 < x_2 = x_3$. Hence, we assume that $x_1 < x_2 < x_3$.

We make a further simplification that $x_1 < 0, x_2 = 0, x_3 = 1$ for the following reason: If we subtract x_2 from each of the first three terms, then stability and length are unaffected. Next, if we divide each by the maximum of the three new numbers, then we attain the aforementioned format, where once again stability and length are unaffected. In other words, given any M&m sequence $\mu(\sigma) = (x_n : n \in \omega)$, where $x_1 < x_2 < x_3$, there exists a corresponding M&m sequence $\mu(\sigma') = (y_n : n \in \omega)$, where $y_1 < 0, y_2 = 0, y_3 = 1$, such that $\mu(\sigma)$ and $\mu(\sigma')$ have the same stability and length. Returning to our first example again, we started with $\sigma = (-7, 25, 29)$. If we subtract the median from each, then we get $(-32, 0, 4)$. Next, we divide each by the maximum of these three new numbers (4) to get $\sigma' = (-8, 0, 1)$. Note that

$$\mu(\sigma') = (-8, 0, 1, 7, 2.5, 3.5, 6.25, 7.75, 7, 8, 18.625, 21.375, 11.125, 11.875, 7, 7, 7...)$$

is stable with length 15, just as we expected. Further, the stable value is exactly what we would expect it to be:

$$\text{stable value of } \mu(\sigma') = \frac{\text{stable value of } \mu(\sigma) - \text{ median of } \sigma}{\text{maximum of new numbers}} \iff 7 = \frac{53 - 25}{4}.$$

We can suppose that $x_1 \leq -1$ because otherwise we may divide each term by x_1 to obtain $\sigma = (\frac{1}{x_1}, 0, 1)$, where $\frac{1}{x_1} \leq -1$ because $-1 \leq x_1 < 0$.

Let $\sigma_t = (-t, 0, 1)$ where $t > 1$. For each $t > 1$, we define a function $f(t)$ as follows: If $\mu(\sigma_t)$ is stable, then $f(t)$ denotes the stable value. Otherwise, $f(t)$ is undefined. The problem of determining whether or not all M&m sequences are stable has now been reduced to determining whether or not f is defined for all $t > 1$. We can make a further restriction that $t \geq 2$ for the following reason: If we start with $\sigma_s = (-s, 0, 1)$ where $1 < s < 2$, then we know that the fourth term, x_4, will be $s - 1$. Since reordering x_1, x_2, x_3, x_4 will not change the fact that $m_4 = \dfrac{s-1}{2}$, we may reorder the four terms without affecting the stability. Keeping this in mind, we can divide each term by $s - 1$ and reorder them to obtain $(-[\dfrac{s}{s-1}], 0, 1, \dfrac{1}{s-1})$. Now we can make the substitution $t = \dfrac{s}{s-1}$ to obtain the sequence $(-t, 0, 1, t-1)$, where $t \geq 2$.

There are a few results proved in the paper by Shultz and Shiflett [?], which we now restate (in our notation) and prove prior to proceeding.

Theorem 1. *The terms of $\mu(\sigma_t)$ are given by*

$$x_4 = t - 1,$$

and for $n \geq 5$,

$$x_n = nm_{n-1} - (n-1)m_{n-2}.$$

Proof. Since

$$M_4 = \frac{-t + 0 + 1 + x_4}{4} = m_3 = 0,$$

we have $x_4 = 0 - (-t + 1) = t - 1$. Now, let $n \geq 5$. By the definition of a mean,

$$M_n = \frac{x_1 + x_2 + ... + x_n}{n} \qquad \text{and} \qquad M_{n-1} = \frac{x_1 + x_2 + ... + x_{n-1}}{n-1}$$

and so

$$nM_n = x_1 + x_2 + ... + x_n \qquad \text{and} \qquad (n-1)M_{n-1} = x_1 + x_2 + ... + x_{n-1}.$$

Note that

$$\begin{aligned} x_n &= (x_1 + x_2 + ... + x_n) - (x_1 + x_2 + ... + x_{n-1}) \\ &= nM_n - (n-1)M_{n-1} \end{aligned}$$

and by the nature of M&m sequences we have $M_n = m_{n-1}$ and $M_{n-1} = m_{n-2}$. It follows that $x_n = nm_{n-1} - (n-1)m_{n-2}$. This completes the proof. □

Another important theorem in [?] relating the sequence values and the medians is the following:

Theorem 2. *If* $\langle x_n \rangle = \mu(\sigma_t)$, *then*

$$x_n \geq m_{n-1}$$

and

$$m_n \geq m_{n-1}$$

for all $n \geq 2$.

Proof. We have

$$x_3 = 1 > 0 = x_2 = m_3 > \frac{-t}{2} = m_2 > -t = m_1$$

and

$$x_4 = t - 1 \geq 1 > 0 = m_3.$$

Since $x_3 = 1$ and $x_4 = t - 1$, which are both nonnegative, we must have

$$m_4 = \frac{0 + 1}{2} = \frac{1}{2} > 0 = m_3.$$

For induction, let $k \geq 5$, and assume $x_{k-1} \geq m_{k-2}$ and $m_{k-1} \geq m_{k-2}$. Then,

$$x_k = km_{k-1} - (k-1)m_{k-2} = m_{k-1} + (k-1)(m_{k-1} - m_{k-2}) \geq m_{k-1}.$$

Hence $x_k \geq m_{k-1}$. So, as we move from $(x_1, x_2, ..., x_{k-1})$ to $(x_1, x_2, ..., x_k)$, the new term x_k is greater or equal to the previous median m_{k-1}. So the new median m_k cannot be smaller than the previous median m_{k-1}. Thus, $m_k \geq m_{k-1}$. By the Principle of Mathematical Induction, this completes the proof. □

One final important theorem from [**?**] restricts our attention to an even smaller interval for t.

Theorem 3. *If* $t \geq 21.3125 = \frac{325}{16}$, *then* $\mu(\sigma_t)$ *is stable with length* 73 *and stable value* 20.3125.

So, now our interval of concern for t has been reduced to $[2, 21.3125)$. As far as stability is concerned, we have not yet been able to prove or disprove the *M&m Conjecture.* However, there are a few things about these sequences that we can prove to extend the results of Shultz and Shiflett. First, we will prove a few propositions that were stated in the paper by Shultz and Shiflett without proof.

Proposition 1. $\mu(\sigma_t)$ *is constant at the* k^{th} *term if and only if* $m_{k-1} = m_{k-2}$.

Proof. First suppose that the sequence is constant at the k^{th} term, i.e. $x_k = x_{k+1}$. Suppose to the contrary that $m_{k-1} \neq m_{k-2}$. By Theorem 2, the sequence of medians is a nondecreasing sequence and so $m_{k-1} > m_{k-2}$. By Theorem 1, we get that

$$\begin{aligned} x_k &= km_{k-1} - (k-1)m_{k-2} \\ &> km_{k-1} - (k-1)m_{k-1} \\ &= m_{k-1}, \end{aligned}$$

which implies that

$$\begin{aligned} x_{k+1} &= (k+1)m_k - km_{k-1} \\ &< (k+1)m_k - kx_k \\ &= (k+1)m_k - kx_{k+1}. \end{aligned}$$

From this it follows that $(k+1)x_{k+1} < (k+1)m_k$ and so $x_{k+1} < m_k$, which contradicts Theorem 2. Hence $m_{k-1} = m_{k-2}$.

Now suppose that $m_{k-1} = m_{k-2}$. From Theorem 1,

$$\begin{aligned} x_k &= km_{k-1} - (k-1)m_{k-2} \\ &= km_{k-1} - (k-1)m_{k-1} \\ &= m_{k-1}. \end{aligned}$$

Since $x_k = m_{k-1}$, then it is easy to see that $m_k = m_{k-1}$, which yields that $x_k = m_k$. It follows that

$$\begin{aligned} x_{k+1} &= (k+1)m_k - km_{k-1} \\ &= (k+1)x_k - kx_k \\ &= x_k. \end{aligned}$$

So, the sequence is constant at the k^{th} term. This completes the proof. □

The next proposition is an extension of Theorem 2.

Proposition 2. *Let $\mu(\sigma_t) = \langle x_n \rangle$. If $\mu(\sigma_t)$ is stable, then the median sequence is strictly increasing and $x_n > m_{n-1}$ until stability is reached. Otherwise, the median sequence is always strictly increasing and $x_n > m_{n-1}$ for all $n \in \mathbb{N}$.*

Proof. Using Theorem 2, it suffices to show that $m_n \neq m_{n-1}$ and $x_n \neq m_{n-1}$ until stability is reached. By Proposition 1, we know that $m_n \neq m_{n-1}$ until stability is reached. Thus, we only need to show that $x_n \neq m_{n-1}$ until stability is reached. Suppose that the sequence is not stable at step n. So $x_n \neq x_{n-1}$. From Proposition 1, we know that $m_{n-1} \neq m_{n-2}$. Hence,

$$\begin{aligned} x_n &= nm_{n-1} - (n-1)m_{n-2} \\ &\neq nm_{n-1} - (n-1)m_{n-1} \\ &= m_{n-1}. \end{aligned}$$

Now suppose that the sequence reaches stability at step n. So, $x_n = x_{n+1}$ and by Proposition 1 we have $m_{n-1} = m_{n-2}$. It follows that

$$\begin{aligned} x_n &= nm_{n-1} - (n-1)m_{n-2} \\ &= nm_{n-1} - (n-1)m_{n-1} \\ &= m_{n-1}. \end{aligned}$$

This completes the proof. □

Proposition 3. *If $\mu(\sigma_t)$ is stable, then its length is the smallest odd number k for which $m_{k-1} = m_{k-2}$.*

Proof. Suppose that $\mu(\sigma_t) = \langle x_n \rangle$ is stable. The length of the sequence is defined to be the smallest k for which $x_n = x_k$ for each $n \geq k$. Let k be the length of $\mu(\sigma_t)$. So, by definition, $x_n = x_k$ for each $n \geq k$, and in particular, $x_k = x_{k+1}$, which means that $\mu(\sigma_t)$ is constant at step k. By Proposition 1, we know that $m_{k-1} = m_{k-2}$. So, k is a number for which $m_{k-1} = m_{k-2}$. Now we need to show that k is odd and that k is the smallest such odd number. Suppose to the contrary that k is even. We showed in our proof of Proposition 2 that since $\mu(\sigma_t)$ is stable at step k, then $x_k = m_{k-1} = m_{k-2}$. Since k is even, $k-1$ is odd and so $m_{k-1} = x_k$ is a term in the finite sequence $(x_1, x_2, ..., x_{k-1})$, i.e., $x_k \in \{x_1, x_2, ..., x_{k-1}\}$. Also, $k-2$ is even and so $m_{k-2} = x_k$ is the average of the two middle numbers[2] in the finite sequence $(x_1, x_2, ..., x_{k-2})$. Say $x_k = \dfrac{x_i + x_j}{2}$, where $i, j \in \{1, 2, ..., k-2\}$ and $i < j$. If $x_i = x_j = x_k$, then $m_{k-3} = x_k$ and it follows that $m_{k-2} = m_{k-3}$. In this case, we know from Proposition 1 that $\mu(\sigma_t)$ is stable at step $k-1$, a contradiction to the fact that $\mu(\sigma_t)$ has length k. Otherwise, we must have $x_i < x_k < x_j$, i.e., x_i and x_j are the two middle numbers in the finite sequence $(x_1, x_2, ..., x_{k-2})$ and they are distinct. This means that $x_k \notin \{x_1, x_2, ..., x_{k-2}\}$. Since $m_{k-1} = x_k$ and $x_k \notin \{x_1, x_2, ..., x_{k-2}\}$, then $x_{k-1} = x_k$. Hence $\mu(\sigma_t)$ is constant at step $k-1$, another contradiction to the fact that $\mu(\sigma_t)$ has length k. Hence k is odd. Now suppose to the contrary that there exists an odd number $l < k$ such that $m_{l-1} = m_{l-2}$. By Proposition 1, $\mu(\sigma_t)$ is constant at step l and so $\mu(\sigma_t)$ has length $l < k$, which contradicts our assumption that $\mu(\sigma_t)$ has length k. This completes the proof. □

Recall the function $f(t)$ that we constructed earlier. We have narrowed the area of interest to $t \in [2, 21.3125)$. However, we can say a few more things about the behavior of the function f.

Theorem 4. *For all $2 \leq t < 21.3125$ for which f is defined, $f(t) \geq t - 1$.*

Proof. Through a tedious process of comparisons of linear functions of t done by hand and recorded onto a spreadsheet, we find that if $2 \leq t < 21.3125$, then $t-1$ will show up as an element in the list of medians when we construct $\mu(\sigma_t)$. By Theorem 2, this list of medians is a nondecreasing sequence. Hence, if f is defined at $2 \leq t < 21.3125$, then $f(t) \geq t - 1$.[3] □

We made a few other observations about M&m sequences via computer programs written in MATLAB. Let $s = 21.3125$ and consider $\mu(\sigma_s)$. We know by Theorem 3 that $\mu(\sigma_s)$ has length 73 and a stable value of 20.3125. Let

[2] Note that 'middle numbers' in this context refers to the middle of the sorted list of this finite sequence.

[3] Note: this may also be proved by observing the sequence $\mu(\sigma)$, where $\sigma = (-21.3125, 0, 1)$, along with its sequence of medians.

$A = \{m_3 + 1, m_5 + 1, ..., m_{69} + 1, m_{71} + 1\}$, i.e., the collection of odd numbered medians found in the construction of $\mu(\sigma_s)$, each summed with 1.[4]

Observation. *If $t \in A$, then $f(t) = t - 1$.*

Another noteworthy pattern emerged from observing the behavior of f near these values of A. Let $t \in A$. So, $f(t) = t - 1$ and we know that $t - 1$ is a minimum for the function $f(t)$. There exist two intervals, $L_t = [\alpha_t, t]$ and $R_t = [t, \beta_t]$, where $\alpha_t < t < \beta_t$, such that f is a linear function on L_t and R_t. Let these two linear functions on L_t and R_t be $a_L t + b_L$ and $a_R t + b_R$, respectively. It turns out that $a_L + a_R = 1$ and $b_L + b_R = t - 2$.

Finally, we used a MATLAB program to determine intervals of stability. The program accepts a particular value, t_0, and a direction, L or R, from the user. It then returns[5] a new value, t_1, and an iteration number i_{t_0}. If the user inputs L, then $t_1 < t_0$ and for each $t \in [t_1, t_0]$, $\mu(\sigma_t)$ is stable with length i_{t_0}. If the user inputs R, then $t_0 < t_1$ and for each $t \in [t_0, t_1]$, $\mu(\sigma_t)$ is stable with length i_{t_0}. The program can be called repeatedly using the previous output as the next input to continually extend the initial interval of stability. The intention of the program was to determine stability on as much of the interval $[2, 21.3125)$ as possible. The program was highly successful in some areas and quite unsuccessful in other areas. For instance, we found a single interval of length 3/2, namely $[15.95, 17.45]$, on which f is constant (stable value 20.3125), using only a few seconds of computer output. However, when we entered $t_0 = 13$ accompanied with either direction, after several hours of computer output, we only found an extremely small interval of length 9.38×10^{-20}, on which f is constant.

So, aside from the obvious open question of whether or not all M&m are stable, there are a few remaining open questions concerning the behavior of f.

1. Are all M&m sequences convergent?
2. If f is defined at t_0, is there an interval, I_{t_0}, of positive length, where $t_0 \in I_{t_0}$, such that f is linear on I_{t_0}?
3. If the M&m conjecture is true, is f a continuous function?
4. For what values of $t \notin A$ (if any) do we have $f(t) = t - 1$?

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[4] Obviously, we are omitting $m_1 + 1$ from this list because m_1 doesn't make sense in this context.

[5] Here we are assuming that $f(t)$ is actually defined and the program ran without errors.

References

[1] Shultz, H. and Shiflett, R. "M&m Sequences." *The College Mathematics Journal,* **Volume** 36, Number 3, (May 2005).