Arc Length and Surface Area: Are Textbooks on the Same Page?

June 12, 2008

An element of arc length or an element of surface area can naturally be approximated using either an inscribed object or a tangential object. An approach often followed is to use inscribed objects for arc length and tangential objects for surface area. Why the difference? This paper explores the roads less traveled.

Both inscribed and tangential approximations can be used in either situation - arc length or surface area. On one hand, the proofs for the inscribed polygonal approach involve deep mathematical analysis, but the approach is more intuitive. On the other hand, the tangential approaches have easier proofs at the expense of some intuition. Also, there is a paradoxical result with a certain polygonal approach for surface area, which will be discussed. In order to appreciate the less traveled approaches, a review of the customary approaches is recommended (see [3] for examples).

The less traveled tangential approach to arc length does not require the deep mathematical analysis (like the use of the Mean Value Theorem when using inscribed objects), and hence its proof is considerably less complicated. Let C be defined by $y = f(x)$, for $a \leq x \leq b$, and assume f' is continuous. A tangential approximation to C is obtained by creating a partition P of the interval $[a, b]$, say $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Let T_i denote the line segment with endpoints (x_{i-1}, y_{i-1}) and (x_i, y_i^*), where T_i is tangent to C at (x_{i-1}, y_{i-1}) (See Figure 1). The sum of the lengths of these segments is approximately the length of C. So, *arc length* is defined as

$$L(C) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} |T_i|,$$

where $|T_i|$ is the length of T_i. Note that

$$f'(x_{i-1}) = \frac{y_i^* - y_{i-1}}{x_i - x_{i-1}} \qquad \text{and so} \qquad y_i^* - y_{i-1} = f'(x_{i-1})(x_i - x_{i-1}).$$

With a little algebra and geometry, it follows that

$$|T_i| = \sqrt{1 + [f'(x_{i-1})]^2}\, \Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$. So, by the definition of arc length and the definition of the definite integral,

$$L(C) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

This formula is identical to that obtained through the polygonal approach.

An intuitive inscribed object approach can be used to calculate surface area, though this approach is much less traveled, perhaps because it involves nontrivial analysis. Suppose a surface S is defined by $z = f(x, y)$, where the domain of f is the rectangle $R = [a, b] \times [c, d]$, and suppose f has continuous partial derivatives. Following the development of arc length, it seems natural to partition R into subrectangles. Let P be a partition of R, say $P = \{R_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$, where $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $\{a = x_0 < x_1 < \cdots < x_n = b\}$ and $\{c = y_0 < y_1 < \cdots < y_m = d\}$ partitions of $[a, b]$ and $[c, d]$, respectively. For each i and j, let $P_{i-1,j-1} = (x_{i-1}, y_{j-1}, f(x_{i-1}, y_{j-1}))$; so $P_{i-1,j-1}$ is the point on S directly above (x_{i-1}, y_{j-1}). Since a collection of 4 points may fail to be coplanar, it is necessary to further divide each subrectangle R_{ij} into two right triangles. Consider one such triangle in the xy-plane with vertices (x_{i-1}, y_{j-1}), (x_{i-1}, y_j), and (x_i, y_j). Let $P_{i-1,j-1}$, $P_{i-1,j}$, and $P_{i,j}$ be the corresponding points on S, and let T_1 be the triangle in $\mathbb{R}^3$ with these points as vertices (see Figure 2). From vector calculus, the area of this triangle is $A(T_1) = \frac{1}{2}\|\vec{v} \times \vec{w}\|$, where $\vec{v}$ is the vector from $P_{i-1,j-1}$ to $P_{i-1,j}$ and $\vec{w}$ is that from $P_{i-1,j}$ to $P_{i,j}$. Since

$$\begin{aligned} \vec{v} &= P_{i-1,j} - P_{i-1,j-1} = \langle 0, \Delta y_j, f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})\rangle, \\ \vec{w} &= P_{i,j} - P_{i-1,j} = \langle \Delta x_i, 0, f(x_i, y_j) - f(x_{i-1}, y_j)\rangle, \end{aligned}$$

then

$$\|\vec{v}\times\vec{w}\| = \sqrt{1+\left[\frac{f(x_i, y_j) - f(x_{i-1}, y_j)}{x_i - x_{i-1}}\right]^2 + \left[\frac{f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})}{y_j - y_{j-1}}\right]^2}\,\Delta x \Delta y,$$

where $\Delta x = x_i - x_{i-1}$ and $\Delta y = y_j - y_{j-1}$. By the Mean Value Theorem, there exist numbers x_i^* and y_j^* with $x_{i-1} < x_i^* < x_i$ and $y_{j-1} < y_j^* < y_j$ so that

$$\|\vec{v} \times \vec{w}\| = \sqrt{1 + [f_x(x_i^*, y_j)]^2 + [f_y(x_{i-1}, y_j^*)]^2}\,\Delta x \Delta y$$

and thus

$$A(T_1) = \frac{1}{2}\sqrt{1 + [f_x(x_i^*, y_j)]^2 + [f_y(x_{i-1}, y_j^*)]^2}\,\Delta x \Delta y.$$

Repeating the analogous arguments for the triangle T_2 with vertices $P_{i-1,j-1}$, $P_{i,j-1}$, and $P_{i,j}$ yields that

$$A(T_2) = \frac{1}{2}\sqrt{1 + [f_x(x_i^{**}, y_{j-1})]^2 + [f_y(x_i, y_j^{**})]^2}\,\Delta x \Delta y,$$

where $x_{i-1} < x_i^{**} < x_i$ and $y_{j-1} < y_j^{**} < y_j$. The sum of the areas of these two triangles is an approximation to the area of the portion of S to which they correspond. Thus, the sum $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}[A(T_1)+A(T_2)]$ is a polyhedral approximation to the total area of S. So, *surface area* is defined as

$$A(S) = \lim_{\|P\|\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}[A(T_1) + A(T_2)].$$

From this definition of surface area,

$$\begin{aligned} A(S) = \lim_{\|P\|\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{2}\Bigg(&\sqrt{1 + [f_x(x_i^*, y_j)]^2 + [f_y(x_{i-1}, y_j^*)]^2} \\ &+\sqrt{1 + [f_x(x_i^{**}, y_{j-1})]^2 + [f_y(x_i, y_j^{**})]^2}\Bigg)\Delta x_i \Delta y_j \end{aligned} \tag{1}$$

This double sum is not a Riemann sum because the points (x_i^*, y_j), (x_{i-1}, y_j^*), (x_i^{**}, y_{j-1}), and (x_i, y_j^{**}) are not all the same point in R_{ij} for each pair i and j. However, it does resemble a Riemann sum. Since f has continuous partial derivatives, Duhamel's principle can be applied to this limit (see Olmsted [1], where it is stated for functions of two variables). Here the statement has been altered, along with the proof.

Theorem 1. *Let $g, h, k,$ and ℓ be continuous functions on a rectangle R in the plane and let ϕ be everywhere continuous in $\mathbb{R}^4$. Also, let $P = \{R_{ij} : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a partition of R and for $d = 1, 2, 3, 4$, let $({x_i}^d, {y_j}^d)$ be points in R_{ij}. Then, the limit*

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(g(x_i^1, y_j^1), h(x_i^2, y_j^2), k(x_i^3, y_j^3), \ell(x_i^4, y_j^4))\Delta x_i \Delta y_j$$

exists and is equal to

$$\iint_R \phi(g(x, y), h(x, y), k(x, y), \ell(x, y))\, dxdy.$$

Proof. Since $g, h, k,$ and ℓ are continuous on R, a compact set, they are each bounded and uniformly continuous on R. Hence, there exists a number M such that each function lies between $-M$ and M. Since ϕ is continuous on the compact set

$$D = \{(s, t, u, v) : |s| \le M, |t| \le M, |u| \le M, |v| \le M\},$$

ϕ is uniformly continuous on D. So, for a given $\epsilon > 0$, there exists $\eta > 0$ such that

$$\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2 + (u_2 - u_1)^2 + (v_2 - v_1)^2} < \eta$$

implies that

$$|\phi(s_2,t_2,u_2,v_2)-\phi(s_1,t_1,u_1,v_1)|<\frac{\epsilon}{2A(R)}.$$

Also, since $\phi(g,h,k,\ell)$ is integrable, there exists δ_1 so that whenever $\|P\|<\delta_1$ and (x_i^*,y_j^*) is any point in R_{ij}, the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(g(x_i^*,y_j^*),h(x_i^*,y_j^*),k(x_i^*,y_j^*),\ell(x_i^*,y_j^*))\Delta x_i\Delta y_j,$$

is within $\frac{\epsilon}{2}$ of the integral

$$\iint_R \phi(g(x,y),h(x,y),k(x,y),\ell(x,y))\,dx\,dy.$$

Since $h,k,$ and ℓ are uniformly continuous on D, there exists $\delta_2>0$ such that if $\|P\|<\delta_2$, then

$$\left|h(x_i^2,y_j^2)-h(x_i^1,y_j^1)\right|<\frac{\eta}{\sqrt{3}},$$
$$\left|k(x_i^3,y_j^3)-k(x_i^1,y_j^1)\right|<\frac{\eta}{\sqrt{3}},$$

and

$$\left|\ell(x_i^4,y_j^4)-\ell(x_i^1,y_j^1)\right|<\frac{\eta}{\sqrt{3}}.$$

Let $\delta=\min\{\delta_1,\delta_2\}$. Through some straightforward calculations it follows that if $||P||<\delta$, then

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(g(x_i^1,y_j^1),h(x_i^2,y_j^2),k(x_i^3,y_j^3),\ell(x_i^4,y_j^4))\Delta x_i\Delta y_j \right.$$
$$\left. -\iint_R \phi(g(x,y),h(x,y),k(x,y),\ell(x,y))\,dxdy\right|<\epsilon.$$

□

Thus, by Duhamel's principle, the limit in (1) converges to the same integral to which it would converge if it were a Riemann sum. It follows that

$$A(S)=\iint_R\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,dx\,dy,$$

which is identical to that obtained through the tangential approach.

It's important to note that paradoxical results can be obtained in a surface area computation using what seems to be a natural polyhedral approach. In 1890, such an approach was described by Schwarz (1843-1921), when he provided a simple example of a polyhedral approximating method of finding the surface area of a cylinder. Schwarz found that his approximation method, though seemingly consistent with the accepted definition of surface area, gave incorrect results (the details of his analysis are worked out in [4]). The important point is that Schwarz's approach does not use a rectangular partition generated approximation, and furthermore, one cannot extract a rectangular partition from his approach. This additional flexibility enables his approach to generate a paradoxical result, while the polyhedral approach presented here arrives at the expected result. It's also important to note that there are some textbooks that refer to these alternative approaches. In fact, Shanks and Gambill actually go further to mention the Schwarz paradox in their text [2]. However, a large majority of textbooks include neither of these approaches, and therefore give an inscribed arc length approach followed by a tangential surface area approach.

Acknowledgements. The author thanks the College of Arts and Science at Miami University for its support of the work done in this paper under the Dean's Scholar Program during the author's senior year in undergraduate school. The author expresses his sincere appreciation to Mark A. Smith (Miami University), who proposed the topic of this paper and supervised the work done here. The author also thanks Thomas Farmer (Miami University) for very valuable suggestions concerning the exposition, Bill Friel (University of Dayton), who suggested relevant references, V. Frederick Rickey (U.S. Military Academy), who pointed out the Schwarz Paradox, and the referees for providing useful suggestions.

References

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