Distribution vectors and stable state

We consider a population with three age categories, C_n, M_n, S_n. The total population at step n is $P_n = C_n + M_n + S_n$. The distribution vector at step n is

$$D_n = \begin{bmatrix} C_n/P_n \\ M_n/P_n \\ S_n/P_n \end{bmatrix}$$

For example in the situation considered in the previous handout we had

$$B_2 = \begin{bmatrix} 339 \\ 234 \\ 75 \end{bmatrix}$$

with a total population $P_2 = 339 + 234 + 75 = 648$. The distribution vector is

$$D_2 = \begin{bmatrix} 0.52 \\ 0.36 \\ 0.12 \end{bmatrix}$$

The entries of the distribution vector tell us what percentage of the total population is represented by each age population. In this example, 52% of the total population is in the C category, 36% in the M category, and 12% is in the S category. Note that the entries of the distribution vector should add up to 1.

We say that the population has reached a **stable state** at step n if $D_n = D_{n+1}$. That is, if the distribution vector does not change when we go to the next step. The distribution vector D_n achieved when the population reaches a stable state is called the **stable distribution** of the population.

In the example from the previous handout, we have

$$D_2 = \begin{bmatrix} 0.52 \\ 0.36 \\ 0.12 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 0.54 \\ 0.34 \\ 0.12 \end{bmatrix}$$

Since the first two components are not the same, the population has not reached a stable state at step $n = 2$. One can check that the population has reached a stable state at $n = 8$, with stable distribution vector

$$D_8 = D_9 = \begin{bmatrix} 0.536 \\ 0.341 \\ 0.123 \end{bmatrix}$$

Eigenvalues and Eigenvectors of a matrix

Let A be an $n \times n$ matrix and B a $n \times 1$ vector.

We say that the vector B is an *eigenvector* for the matrix A if there exists a constant λ (called the *eigenvalue* associated to the eigenvector B) such that

$$A \cdot B = \lambda B$$

In other words, B is an eigenvector for the matrix A if $A \cdot B$ is a vector proportional to B, and the constant of proportionality is the corresponding eigenvalue.

Examples: Consider

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We have

$$A \cdot B = \begin{bmatrix} -15 \\ 25 \end{bmatrix}$$

This vector is not proportional to B since $-15/5 \neq 25/-5$. Thus B is not an eigenvector for the matrix A.

$$A \cdot C = \begin{bmatrix} 21 \\ 21 \end{bmatrix}$$

This vector is proportional to C: $21/3 = 21/3 = 7$, in other words $A \cdot C = 7C$. The corresponding eigenvalue is $\lambda = 7$.

Returning to the set up of populations with age structure, assume that A is a transition matrix, and B_n denotes the population vector at step n.

Observation: If the population has reached stable state at step n, this means that the vector B_n is an eigenvector for the matrix A.

Indeed, stable state means that the vectors B_n/P_n and B_{n+1}/P_{n+1} coincide, where by B_n/P_n we mean the vector obtained by dividing each entry in B_n by the total population size P_n. Recalling that $B_{n+1} = AB_n$, this equation can be written as

$$\frac{AB_n}{P_{n+1}} = \frac{B_n}{P_n},$$

or equivalently

$$AB_n = \frac{P_{n+1}}{P_n} \cdot B_n$$

which means that B_n is an eigenvector for A with corresponding eigenvalue P_{n+1}/P_n.

Observation: If B is an eigenvector for a matrix A, then $A \cdot B$ is also an eigenvector for A.

This observation shows that if the population has achieved stable state at step n, the stable state will continue at step $n+1$ (because the vector $B_{n+1} = A \cdot B_n$ is also an eigenvector).

In order to justify our observation, assume that $A \cdot B = \lambda B$ where λ is the eigenvalue corresponding to the eigenvector B. In order to see that $A \cdot B$ is also an eigenvector, we need to calculate $A \cdot (A \cdot B)$:

$$A(A \cdot B) = A \cdot (\lambda B) = \lambda(A \cdot B) = \lambda(\lambda B) = \lambda^2 B$$

Thus $A \cdot B$ is an eigenvector with corresponding eigenvalue λ^2.

Observation: Exponential behavior follows once stable state is reached The calculation above shows that if B is an eigenvector for the matrix A with eigenvalue λ (so $A \cdot B = \lambda B$) then $A^n \cdot B = \lambda^n B$.

Thus if we have a population vector B_n that has reached a stable state at step n we have $B_{n+1} = \lambda B_n$, $B_{n+2} = \lambda^2 B_n$, $B_{n+3} = \lambda^3 B_n$, etc.

Component-wise this means that $C_{n+2} = \lambda^2 C_n$, $M_{n+2} = \lambda^2 M_n$, $S_{n+2} = \lambda^2 S_n$, and when we add these numbers we see that $P_{n+2} = \lambda^2 P_n$. Similarly we have $P_{n+3} = \lambda^3 P_n$, etc. which shows that the total population size will exhibit exponential behavior once a stable state is reached.