Math 172 Spring 2011 February 2 Handout

Solving differential equations by separation of varibles and integration by parts

You will need to use the following integrals from calculus:

$$\int dx = x+C \qquad \int a\,dx = ax+C \qquad \int \frac{1}{x}\,dx = \ln|x|+C \qquad \int \frac{1}{rx+c}\,dx = \frac{1}{r}\ln|rx+c|+C$$

In all of the above integrals, C stands for an arbitrary constant. The other letters (a, r, c stand for specified numerical values). We will find the numerical value of the constant C that appears in the integration based on the initial value that will be given in our specific problems. An expression inside the $|\ \ |$ symbol means the absolute value of that expression.

Also the integrals that we will usually deal with will have different names for the variable instead of x (such as t and P). We will use the same letter that appears in the integral when we give the answer for that integral (substituting it for x in the formulas above).

Now we are ready to solve the differential equations for the affine models:

$$\frac{dP}{dt} = rP + c$$

(where the constants r, c could be positive or negative).

First we separate the variables by putting the expression $rP+c$ that contains P on the left hand side, and the expression dt which contains t on the right hand side:

$$\frac{dP}{rP+c} = dt$$

Now we integrate each side and set the two integrals equal to each other:

$$\int \frac{1}{rP+c}\,dP = \int dt$$

The integral on the left hand side is equal to $\frac{1}{r}\ln|rP+c| + C_1$, and the integral on the right hand side is equal to $t + C_2$, where C_1, C_2 are arbitrary constants.

Now we set the two results equal to each other:

$$\frac{1}{r}\ln|rP+c| + C_1 = t + C_2$$

Moving the constant C_1 to the right hand side and using C to denote $C_2 - C_1$, we can write this in a slightly simpler form (with one arbitrary constant instead of two):

$$\frac{1}{r}\ln|rP+c| = t + C \tag{1}$$

The calculus part of our problem is now over. We need to solve from P from the above equation. We only need to use algebra to do this.

We will use the convention to just write C for any other constant that appears throughout the process (even though it is not the same as the original C, but some other arbitrary constant). We will find the numerical value of that constant at the end based on the initial value given in the problem.

First multiply through by r on both sides of the equation 1 above. We get:

$$\ln|rP+c| = rt + C$$

Now exponentiate each side:

$$e^{\ln|rP+c|} = e^{rt+C} = e^C e^{rt}$$

Since the exponential and logarithmic function are inverses to each other, this gives us:

$$|rP+c| = e^C e^{rt} \tag{2}$$

At this point we can find a numerical value for e^C so we won't have to keep working with an arbitrary constant C any more. In order to do this plug in $t=0$ in each side of equation 2 above. On the left we will have $|rP(0)+c|$ and on the right we will have e^C (since $e^0 = 1$). Thus we have

$$e^C = |rP(0)+c|. \tag{3}$$

Now we need to deal with the absolute value. If both r and c are positive, then $|rP+c| = rP+c$, and we can solve for P from equation 2. We obtain:

$$P = \frac{e^C e^{rt} - c}{r}$$

In order to have concrete numerical values replace e^C with the value from equation 3.

In the affine models that we usually study, the value of either r or c is negative; this means that $rP+c$ could be negative and when this is the case $|rP+c| = -(rP+c)$. I will discuss the two possibilities separately.

Assume that r is negative and c is positive. I will write $-r$ instead of r in this case, so the original equation is $\frac{dP}{dt} = -rP + c$. Going back to equation 2, we now have

$$|-rP + c| = e^C e^{-rt} \tag{4}$$

The absolute value $|-rP+c|$ is equal to $-rP+c$ when $P < \frac{c}{r}$ and it is equal to $rP-c$ when $P > \frac{c}{r}$. Thus the formula for the final answer will depend on whether the initial value $P(0)$ is smaller or larger than $\frac{c}{r}$ (recall that $\frac{c}{r}$ is the equilibrium value for P). If $P(0) < \frac{c}{r}$ the formula is:

$$P = \frac{c - e^C e^{-rt}}{r}$$

and if $P(0) < \frac{c}{r}$ the formula is:

$$P = \frac{c + e^C e^{-rt}}{r}$$

Again we can obtain a formula that does not involve an arbitrary C by using 3 (remember to use $-r$ instead of r in 3).

Since the exponential part e^{-rt} approaches zero in the long run, we can see from these formulas that the long term behavior for P is to approach $\frac{c}{r}$ (the equilibrium value).

Now assume that r is positive and c is negative. I will write $-c$ instead of c in this case, so the original equation is $\frac{dP}{dt} = rP - c$. Going back to equation 2, we now have

$$|rP - c| = e^C e^{rt} \tag{5}$$

The absolute value $|rP - c|$ is equal to $rP - c$ when $P > \frac{c}{r}$ and it is equal to $-rP + c$ when $P < \frac{c}{r}$. Thus the formula for the final answer will depend on whether the initial value $P(0)$ is smaller or larger than $\frac{c}{r}$ (the equilibrium value). If $P(0) > \frac{c}{r}$ the formula is:

$$P = \frac{c + e^C e^{rt}}{r}$$

and if $P(0) < \frac{c}{r}$ the formula is:

$$P = \frac{c - e^C e^{rt}}{r}$$

Again we can obtain a formula that does not involve an arbitrary C by using equation 3 (remember to use $-c$ instead of c in equation 3).

Since the exponential part e^{rt} approaches infinity in the long run, we see that P will approach infinity (grow without bound) if $P(0) > \frac{c}{r}$ and will approach negative infinity (since for us P represents a population, negative values don't make sense, so from our point of view this means that the population becomes eventually extinct) if $P(0) < \frac{c}{r}$.