Populations with age structures

Consider the discrete process described below:

A population consists of three age classes: children (C), mature individuals (M), and older individuals (O).

At each step, 50% of children become mature individual and 2% of children die; 30% of mature individuals become old, and 5% of them die; 40% of the old individuals die. Moreover, at each step each pair of mature individuals produces 3 children (thus a total of 3*M/2 children are produced).

This process can be described by the following system of three recursive equations:

$$\begin{aligned} C_{n+1} &= 0.48C_n + 1.5M_n \\ M_{n+1} &= 0.5C_n + 0.65M_n \\ O_{n+1} &= 0.3M_n + 0.6O_n \end{aligned}$$

In matrix form, this can be written as

$$\begin{bmatrix} C_{n+1} \\ M_{n+1} \\ O_{n+1} \end{bmatrix} = \begin{bmatrix} 0.48 & 1.5 & 0 \\ 0.5 & 0.65 & 0 \\ 0 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} C_n \\ M_n \\ O_n \end{bmatrix}$$

The 3×3 (3 rows and 3 columns) matrix

$$A = \begin{bmatrix} 0.48 & 1.5 & 0 \\ 0.5 & 0.65 & 0 \\ 0 & 0.3 & 0.6 \end{bmatrix}$$

is called the **transition matrix**.

You should learn how to enter matrices on your TI-83 or TI-84 calculator and perform matrix operations. Detailed instructions can be found at:

http://www.wscc.cc.tn.us/math/jlaprise/Calculator/MatricesonTI83.pdf

Continuing with our example, let's assume that an initial population vector is given:

$$\begin{bmatrix} C_0 \\ M_0 \\ O_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \\ 0 \end{bmatrix}$$

Let us denote the initial population vector as B. This should be entered on the calculator as a 3×1 matrix (3 rows and 1 column).

We can use the transition matrix in order to calculate the population vector after any given number of steps:

after 1 step: $B_1 = A * B$

after 2 steps: $B_2 = A^2 * B$

after n steps: $B_n = A^n * B$ (these operations can be performed on the calculator).

We obtain the following values:

$$B_1 = \begin{bmatrix} 300 \\ 130 \\ 60 \end{bmatrix} \quad B_2 = \begin{bmatrix} 339 \\ 234.5 \\ 75 \end{bmatrix} \quad B_3 = \begin{bmatrix} 514 \\ 322 \\ 115 \end{bmatrix} \quad B_4 = \begin{bmatrix} 730 \\ 466 \\ 166 \end{bmatrix}$$

Recall that if we simplify our assumptions and just consider a population M consisting entirely of mature individuals that produces $1.5M$ offspring at each time step, we would have an exponential model with $\Delta M = 1.5M$; when written as a recursive equation this is the same as $M_{n+1} = 2.5M_n$.

Main Question: Do populations with age structure behave in an exponential manner? By an exponential behavior we mean a recursive equation of the form $P_{n+1} = (1 + r)P_n$, with a constant value of r.

In order to study this question, let us consider the total size of the population $P_n = C_n + M_n + O_n$. We have the following values:

n	0	1	2	3	4	5	6	7	8
P	200	490	648	951	1362	1957	2809	4033	5789

Notice that $P_1 = 2.5P_0$, $P_2 = 1.3P_1$, $P_3 = 1.47P_2$, $P_4 = 1.432P_3$, $P_5 = 1.437P_4$, $P_6 = 1.4354P_5$, $P_7 = 1.4357P_6$, $P_8 = 1.4354P_7$.

It appears that in the long run we have an exponential behavior: $P_{n+1} = (1 + r)P_n$ with $1 + r \cong 1.435$, so $r \cong 0.435$.

Let us confirm this by choosing a large value of n such as $n = 20$ (if we choose a number much larger than this your calculator won't be able to complete the calculation). Compute the population vectors $B_{20} = A^{20} * B_0$ and $B_{21} = A^{21} * B_0$ and find the total population P_{20} and P_{21} by adding the entries in each one of these vectors.

I get $P_{20} = 442027$, $P_{21} = 634391$, and $P_{21}/P_{20} = 1.435$, which confirms the prediction that in the long run this population will behave in an exponential manner with $r = 0.435$.

Operations with matrices and vectors

The notation A^n above stands for **multiplying** the matrix A times itself n times. Multiplication of matrices is an algbraic operation that we discuss below. In this class, we will multiply a matrix times itself (as in taking a power) and we will also multiply a matrix times a vector.

In general one can multiply a $m \times n$ matrix by a $n \times p$ matrix B and the result will be a $m \times p$ matrix. In particular if one multiplis an $m \times n$ matrix by an n-dimensional vector (which can be viewed as a $n \times 1$ matrix) the result will be a m-dimensional vector.

In order to multiply two matrices $A * B$, it is helpful to visualize the first matrix divided into rows and the second matrix divided into columns. In order to fill in the position in row i and column j of the resulting matrix, one does the dot product of the row i in matrix A and the column j in matrix B. The dot product means that the elements in corresponding positions are multiplied together, and then the results are added.

Here is how to multiply 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$$

Here is how to multiply a 3×3 matrix by a 3-dimensional vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$