

The logistic equation

The logistic equation is a modification of the exponential model which takes into account the limitations of the environment. In the exponential model with $r > 0$ we saw that unlimited growth occurs. There is no equilibrium value other than zero (which is unstable for $r > 0$). In practice, unlimited growth is not usually possible, since the environment in which the population grows can only support a certain number of individuals. This number is called the **carrying capacity** of this environment, and it will be a stable equilibrium value for the logistic equation. The implication of having a stable equilibrium is that the size of the population will always approach the carrying capacity (regardless of whether the initial value is above or below).

The logistic equation is

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right).$$

where $r > 0$ is the intrinsic per capita growth rate (the rate at which the population would keep growing in an unlimited environment) and K is the carrying capacity (i.e. the maximum number of individuals that the given environment can support). This is called a density dependent growth rate. If we view the per capita growth rate in this equation to be $r(1 - \frac{P}{K})$ we see that this close to r when P is small, but it becomes smaller as P grows, and it becomes negative when $P > K$. Biologically, the explanation is that overcrowding inhibits growth.

Let us find the equilibrium values and analyze them as stable or unstable. To find equilibrium set

$$rP \left(1 - \frac{P}{K} \right) = 0$$

and solve for P . A product is equal to zero if and only if one of the factors is zero. So the equation above implies $P = 0$ or $1 - \frac{P}{K} = 0$. Thus the solutions are $P = 0$ and $P = K$.

In order to analyze whether the equilibrium values are stable or unstable, we study the following chart:

$$\begin{array}{ccccc} P & 0 & & K & \\ \frac{dP}{dt} & 0 & + & 0 & - \end{array}$$

The explanation for the signs in the chart is as follows: When the value of P is between 0 and K , we have $\frac{P}{K} < 1$ so $1 - \frac{P}{K} > 0$. Since P is also positive and multiplying positive quantities gives a positive result,

we conclude that $\frac{dP}{dt} = rP(1 - \frac{P}{K}) > 0$ and this gives us the + in the chart. On the other hand, when $P > K$, $\frac{P}{K} > 1$ and so $1 - \frac{P}{K} < 0$. Also, P is positive, and multiplying a positive with a negative gives a negative result. This explains the - in the chart. The signs in the chart indicate that zero is an unstable equilibrium (that is if P starts out larger than zero it will increase - but this time it is not an unbounded increase, just until it reaches the next equilibrium value of K) and K is a stable equilibrium. The long term outcome for a population with growth modeled by a logistic equilibrium is that the population will approach the carrying capacity K (which is the stable equilibrium value).

We will also study a modification of the logistic equation, which we will refer to as the logistic equation with Allee effect. The Allee effect is the principle that individuals within a population require the presence of other individuals in order to survive and reproduce successfully. Thus when the population size is too small, it will not be able to maintain a positive growth rate.

The logistic equation with Allee effect has the form:

$$\frac{dP}{dt} = rP \left(\frac{P}{A} - 1 \right) \left(1 - \frac{P}{K} \right)$$

where r is the intrinsic per capita growth rate, K is the carrying capacity, and A is the minimal size of the population required to survive. We are assuming that $A < K$. We will see that the long term outcome of a population modeled by this equation depends on whether the initial value is above or below the value of A . For this equation one can find three equilibrium values: $P = 0$, $P = A$ and $P = K$. A chart similar to the one shown above shows that $P = 0$ and $P = K$ are stable equilibrium values, and $P = A$ is unstable. If $P_0 < A$ then the population will eventually become extinct. If $P_0 > A$ then the population will approach K in the long run.

We will not be concerned with finding the explicit solution for the logistic equation (the calculation could be carried out using the method of separation of variables, but we will omit it). Instead, we will focus on a qualitative analysis of the solution.

The main questions we will focus on will be: is the size of the population increasing or decreasing, and if so, how fast is it increasing or decreasing (in other words, is the graph of P concave up or concave down?).

If one graphs $\frac{dP}{dt}$ as a function of P (using the equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$), the graph is an upside down parabola with the vertex at $P = K/2$. This shows that $\frac{dP}{dt}$ is increasing when $P < \frac{K}{2}$ and decreasing when

$$P > \frac{K}{2}$$

. In terms of P : the graph of P is increasing and concave up when $P < \frac{K}{2}$ and increasing and concave down when $P > \frac{K}{2}$. If $P_0 < \frac{K}{2}$, then the graph will have a concave up portion followed by a concave down portion, with the inflection point when $P = \frac{K}{2}$. If $P_0 > \frac{K}{2}$, the whole graph is increasing and concave down.

In order to analyze the behavior of P if $P_0 > K$ consider the portion of the parabola described above which corresponds to values of P that are greater than K . The parabola will then be below the horizontal axis (indicating negative values for $\frac{dP}{dt}$) and it will be getting lower and lower as P gets bigger. But we know that as time goes on P will be getting smaller, not bigger (due to $\frac{dP}{dt}$ being negative), which means that the values of $\frac{dP}{dt}$ will be getting closer and closer to zero. Thus P is decreasing at a slower and slower rate and the graph of P is decreasing and concave up.