

Math 172 Spring 2011 Handout 5; January 24

How to solve discrete affine equations:

The process of finding the general solution for an affine equation of the form $\Delta P = rP - c$ or $\Delta P = -rP + c$ relies on the observation that the deviation of P from the equilibrium value $P_{eq} = c/r$ behaves like an exponential function.

Consider a new function $Q = P - \frac{c}{r}$. We want to establish a difference equation for the function Q .

Let's assume that the original equation for P is $\Delta P = rP - c$ (the other case is similar). We can write $P = Q + \frac{c}{r}$. Then

$$\Delta P = P(t+1) - P(t) = \left(Q(t+1) + \frac{c}{r}\right) - \left(Q(t) + \frac{c}{r}\right) = Q(t+1) - Q(t).$$

Thus, $\Delta P = \Delta Q$. If we also plug in $P = Q + \frac{c}{r}$ in the right hand side of the original equation, we will obtain a difference equation satisfied by the new function Q :

$$\Delta Q = r\left(Q + \frac{c}{r}\right) - c = rQ + r\frac{c}{r} - c = rQ.$$

Thus Q satisfies the difference equation $\Delta Q = rQ$, which tells us that the general formula for Q is the exponential function

$$Q = Q_0(1+r)^t$$

where $Q_0 = P_0 - \frac{c}{r}$.

Once we know the general formula for Q it is easy to go back and find the general formula for P , using $P = Q + \frac{c}{r}$. Thus:

$$P = \left(P_0 - \frac{c}{r}\right)(1+r)^t + \frac{c}{r}$$

This formula also allows us to figure out the long term behavior of $P(t)$:

If r is positive, then $(1+r)^t$ increases indefinitely. In this case, the long term outcome for $P(t)$ depends on whether the initial value P_0 is larger or smaller than the equilibrium value.

If $P_0 > \frac{c}{r}$, then the term $\left(P_0 - \frac{c}{r}\right)(1+r)^t$ in the general formula for P is positive and increasing indefinitely thus making P increase indefinitely.

If $P_0 < \frac{c}{r}$, then the factor $\left(P_0 - \frac{c}{r}\right)$ is negative, and thus $P(t)$ will be decreasing to eventual extinction.

On the other hand if $r < 0$ we have $(1 + r)^t$ approaching zero, and $P(t)$ will approach its equilibrium value regardless of what the initial value is.