Isoclines and the state space

Consider the system of Lotka-Volterra equations for two population in competition:

$$\begin{cases} \frac{dN_1}{dt} &= r_1 N_1 \frac{K_1 - N_1 - \alpha N_2}{K_1} \\ \frac{dN_2}{dt} &= r_2 N_2 \frac{K_2 - N_2 - \beta N_1}{K_2} \end{cases}$$

The **state space** associated to this system is a two-dimensional plot in which the values of N_1 are represented on the horizontal axis and the values of N_2 are represented on the vertical axis.

The state space should show the two **isoclines** and the position of the initial value point; the direction in which the system will evolve should be represented by an arrow which shows the short-term behavior of the system: arrow pointing left means N_1 decrease, arrow pointing right means N_1 increases, arrow pointing down means N_2 decreases, arrow pointing up means N_2 increases.

The direction of the arrow will be determined by the position of the initial point relative to the isocline. The arrow should point toward the isocline. This means that if the initial point is to the right of the isocline of N_1 then arrow points left; if initial point is to the left of the isocline of N_1 then arrow points right. Similarly if the initial point is above isocline of N_2 then arrow points down, and if initial point is below isocline of N_2 then arrow points up.

Let us consider a concrete example. Say $K_1 = 100, K_2 = 150, \alpha = 2, \beta = 3$.

The isocline for N_1 is $N_1 + 2N_2 = 100$. On the plot this is the line that joins the points $(100, 0)$ on the horizontal axis and $(0, 50)$ on the vertical axis.

The isocline for N_2 is $3N_1 + N_2 = 150$. This is the line that joins the points $(50, 0)$ on the horizontal axis, and $(0, 150)$ on the vertical axis.

The two isoclines intersect at the point $(N_1 = 40, N_2 = 30)$. This is the equilibrium point with both populations present. The other equilibria are at $(0, 0)$, $(N_1 = 100, N_2 = 0)$, $(N_1 = 0, N_2 = 150)$.

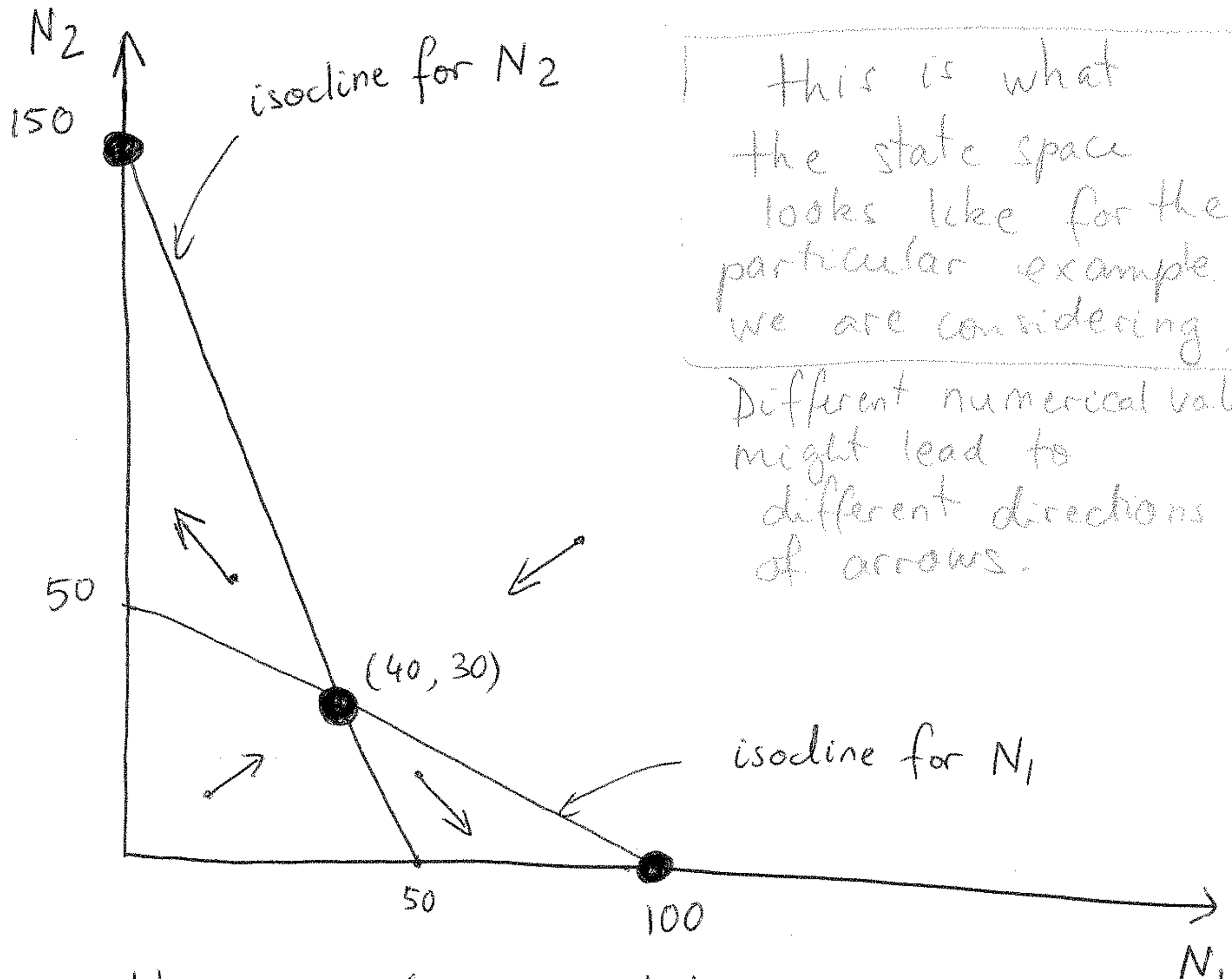
Given an initial value point, we wish to desribe the short term dynamics (this means: for each of N_1, N_2 state whether increasing or decreasing) and (when possible) predict the long term outcome of the competition. We should also be able to decide if the equilibria are stable or unstable based on the direction of the arrows.

Let us consider the initial value point $(N_1 = 25, N_2 = 50)$. We plug these values in each of the isocline equations in order to see what is the position of the initial point relative to the isoclines. We have $N_1 + 2N_2 > 100$ so the point is to the right of the isocline of N_1. This means $\dfrac{dN_1}{dt} < 0$ so the arrow will point to the left. Also $3N_1 + N_2 < 150$. This means that the initial point is below the isocline of N_2, so $\dfrac{dN_2}{dt} > 0$ so the arrow will point up. See the picture on the next page.

Short term behavior: N_1 decreases, N_2 increases. Long term prediction: species 2 will drive species 1 out. Equilibrium $(N_1 = 0, N_2 = 150)$ is approaced. The equilibrium $(N_1 = 40, N_2 = 30)$ is unstable. The equilibrium $(N_1 = 0, N_2 = 150)$ is stable.

In order to make sure that our long term prediction is accurate one should also consider initial value points in the other three regions of the plane. Keep in mind that as the system evolves it might cross from one region into another. In practice I will not require you to do this for all four regions since it is time consuming.

Let us also consider a point in the region of the plane that is above both isoclines, for example $N_1 = 60, N_2 = 70$. We have $\frac{dN_1}{dt} < 0, \frac{dN_2}{dt} < 0$ so the arrow points to the left and down (meaning that both N_1, N_2 decrease). It would appear on the plot that the equilibrium $(N_1 = 40, N_2 = 30)$ is approached. However, as the system evolves the point (N_1, N_2) might move into one of the other regions, and then the short term behavior will change causing one of the populations to increase and the other to decrease. The long term outcome cannot be predicted in this case (it depends on the region that the point will cross into).



this is what the state space looks like for the particular example we are considering.

Different numerical values might lead to different directions of arrows.

equilibria:

(0, 0) unstable

(40, 30) unstable

(100, 0) stable

(0, 150) stable