

**Math 172 Spring 2011 Handout 1: Discrete and
continuous models**

INTRODUCTION AND TERMINOLOGY

We will model the size of a population, P , as a function of time, t . Thus, $P = P(t)$. Sometimes we might use N instead of P , or n instead of t .

In a **discrete model** the time t is measured at discrete intervals, e.g. year 1, year 2, \dots , or day 1, day 2, \dots , depending on what units are being used. In this case the discrete variable t only takes on non-negative integer values, $t = 0, 1, 2, \dots$. Sometimes we use n instead of t in a discrete model.

We are concerned with the **change** in the size of a population. In a discrete model, this is denoted ΔP .

$$\Delta P = P(n + 1) - P(n).$$

An equation that gives information about ΔP is called a **difference equation**, or a **dynamic model**. When we are able to, we will use the difference equation in order to find an explicit formula for $P(t)$. This explicit formula is called the **explicit solution** to the difference equation (also called a **static model**).

We can always translate the difference equation into a **recursive equation**, which tells us how to find $P(n + 1)$ once $P(n)$ is known:

$$P(n + 1) = P(n) + \Delta P.$$

EXAMPLES

Discrete linear model: ΔP is constant.

For example $\Delta P = 3$. This means that for each time unit that passes, the size of the population increases by 3 individuals. In order to be able to find the explicit solution, $P = P(t)$, one also needs to know the **initial value** of the population, P_0 . Let's assume $P_0 = 100$, so the population starts out with 100 individuals, and increases by 3 individuals per unit of time. Let's say that time is measured in days.

We want to find the explicit solution, that is: how many individuals are there after t days?

We have:

$$P(0) = 100, P(1) = 100 + 3 = 103, P(2) = 103 + 3 = 106, \dots$$

In general, after t years, $3 \times t$ individuals will be added to the initial population of 100, so $P(t) = 100 + 3t$ is the explicit solution. Note that the general solution is given by a **linear function**; if we graph P as a

function of t , the graph will be a line with slope equal to 3 and vertical intercept equal to 100.

Discrete exponential model: ΔP is proportional to the value of P . This is usually the case when the change in the population is due to unrestricted reproduction, that is: each reproductive pair gives rise to a fixed number of offspring, let's say 6 (for example) per year.

Let's make the simplifying assumption that the entire population consists of reproductive adults (and the new offspring become reproductive adults as soon as they are born - this is of course not realistic - we will later introduce the concept of populations with age structure in order to make this model more realistic).

Thus, if the total size of the population is P , we have $\frac{1}{2}P$ pairs of reproductive adults, which give rise to $\frac{1}{2}P \times 6 = 3P$ new offspring per year.

The difference equation in this example is:

$$\Delta P = 3P.$$

The recursive equation is

$$P(t+1) = P(t) + 3P(t).$$

Note that as the size of P increases, so does the value of ΔP . Thus, this population will increase faster and faster. This is a trademark of the exponential function.

Let us find the explicit solution corresponding to this difference equation. Again, let's assume that the initial population is $P_0 = 100$.

For the first year, we have $\Delta P = 300$, so $P(1) = P(0) + \Delta P = 100 + 300 = 400$. For the second year, $\Delta P = 3 \times 400 = 1200$, so $P(2) = P(1) + 1200 = 1600$, etc.

In order to write down the general formula, note that the recursive equation (see above) can be written as

$$P(t+1) = 4P(t).$$

Thus, the size of the population gets multiplied by 4 with each passing year. After t years, the size of the population gets multiplied by 4^t times, that is, it gets multiplied by 4^t , giving us the general formula:

$$P(t) = P_0 \times 4^t = 100 \times 4^t$$

As expected, the general formula is given by an **exponential function** (the presence of the variable t in the exponent indicates that this is an exponential function).

(please note: 100×4^t is not the same as 400^t)