Qualitative analysis of the logistic model

The following are possible questions regarding the logistic model that could be asked on the first exam:

1. Assume that a population P has per capita birth and death rates given by $b' = b - aP$, $d' = d + cP$, where a, b, c, d are constants. Explain how to write the equation that models the growth of this population in the form

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

What are the values of r and K in terms of a, b, c, d?

Explain the meaning of the values r and K in biological terms.

Solution:

$$\frac{dP}{dt} = (b' - d')P = (b - aP - d - cP)P = (b - d - (a + c)P)P =$$

$$(b - d)\left(1 - \frac{a + c}{b - d}P\right)P = r\left(1 - \frac{P}{K}\right)P$$

where $r = b - d$, $K = \dfrac{a + c}{b - d}$.

The value of r is the intrinsic growth rate at which the population would grow in the presence of unlimited resources.

The value of K is the carrying capacity of the environment; it is the maximum size of the population that can be supported in the long run by the given environment.

2. Given a logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

find the equilibrium values for P, and for each equilibrium value decide whether it's stable or unstable.

What does this tell us about the long term behavior of the population?

Answer: The equilibrium values are obtained by solving for P in the equation

$$rP\left(1 - \frac{P}{K}\right) = 0;$$

they are $P = 0$ and $P = K$.

The equilibrium value $P = 0$ is unstable: If the initial value of P is a little more than 0 (so positive but less than K) then $/dt > 0$, so the values of P move away from 0.

The equilibrium value $P = K$ is stable: If the initial value of P is less than K then the factor $(1 - P/K)$ is positive, thus dP/dt is positive. This implies that P is increasing, moving towards K.

Also if the initial value of P is larger than K then the factor $(1-P/K)$ is negative, thus dP/dt is negative. This implies that P is decreasing, moving towards K.

In the long run the values of P move towards the carrying capacity K, regardless of the inital value.

3. Consider a logistic model with equation

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{100}\right)$$

. Sketch the graph of $P = P(t)$ for each of the following cases:

a. $P(0) = 10$

b. $P(0) = 80$

c. $P(0) = 150$.

Solution: There are two features of the graph that I will be looking for:

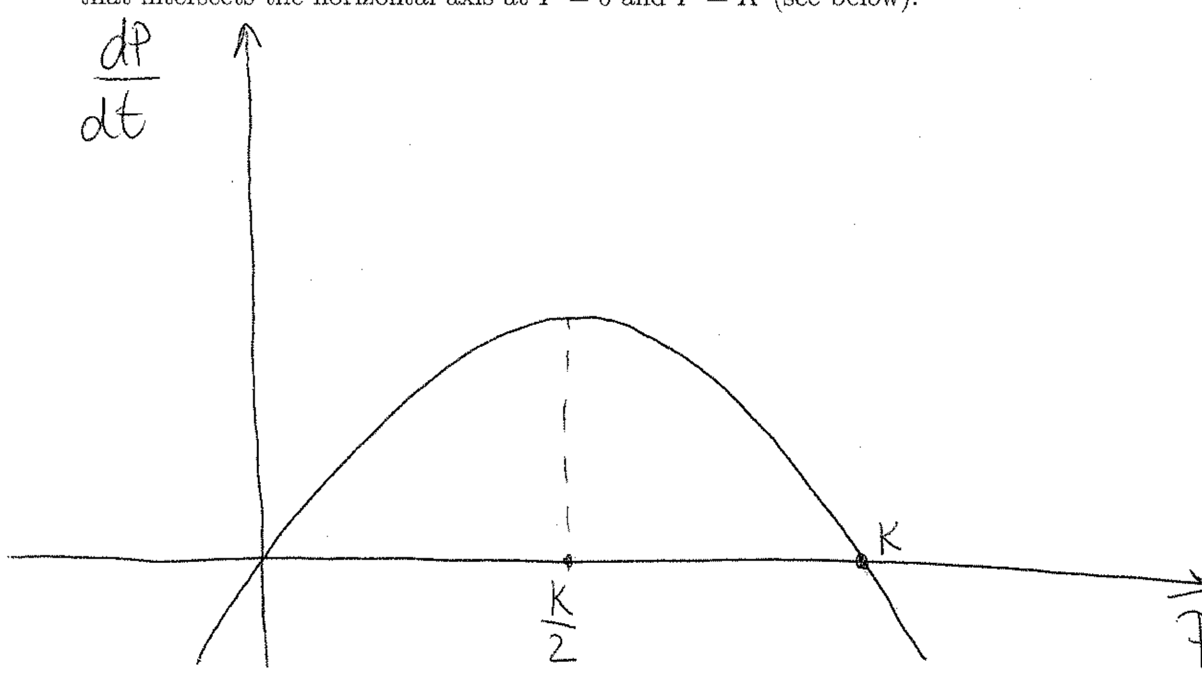
increasing/decreasing, and concave up/concave down

Recall that for an increasing graph, concave up means that the function is increasing faster and faster, while concave down means that the function is increasing slower and slower.

For a decreasing graph, concave up means that the function is decreasing slower and slower, and concave down means that it's decreasing faster and faster.

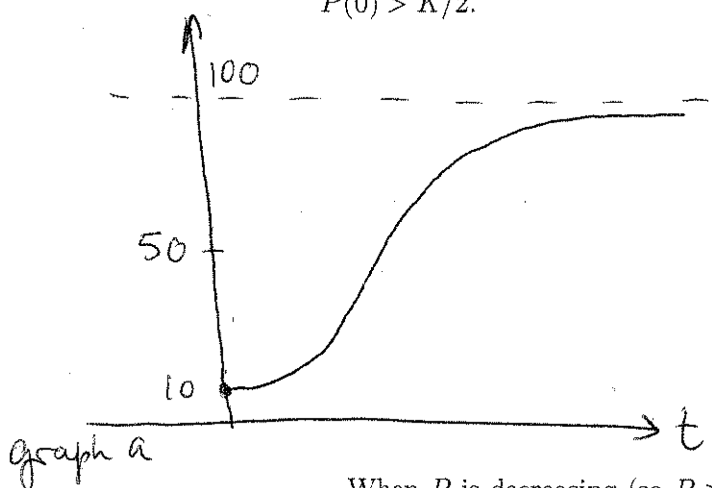
The given equation has carrying capacity $K = 100$. We have seen in question 2. above that $P = P(t)$ is increasing if $P < 100$ and it is decreasing if $P > 100$. Thus the graphs for a. and b. are increasing, and the graph for c. is decreasing.

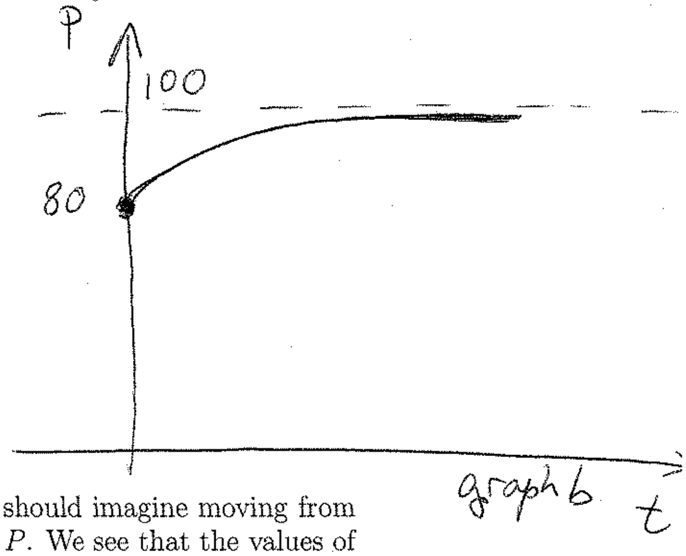
In order to establish whether concave up/concave down, we consider the graph of $\frac{dP}{dt}$ as a function of P, which is an upside-down parabola that intersects the horizontal axis at $P = 0$ and $P = K$ (see below).



When P is increasing, concave up means that the growth rate dP/dt is getting larger and larger. As seen in the graph above, this is the case for $P \leq K/2$. When P is between $K/2$ and K, the growth rate dP/dt, while still positive, is getting smaller and smaller. Thus the graph becomes concave down when $P > K/2$.

Depending on the initial value, a graph with $P(0) < K$ (such as the ones for parts a. and b. of the question) will have either a concave up portion (if $P(0) < K/s$) in the beginning, followed by a concave down portion once P reaches $K/2$, or will be just concave down if $P(0) > K/2$.





When P is decreasing (so $P > K$), we should imagine moving from *right to left* on the graph of dP/dt versus P. We see that the values of dP/dt become closer and closer to zero, so P is decreasing at a slower and slower rate. This means that the graph of P is concave up and decreasing whenever $P(0) > K$.