**Math 300 Fall 2013 Exam 3**

**1.** Use *Rolle's theorem* to show that $x^3 + 5x - 2 = 0$ does not have more than one real solution.

**Rolle's theorem** says that if $y = f(x)$ is a differentiable function, and $x_1 < x_2$ are real numbers such that $f(x_1) = f(x_2) = 0$, then there exists a real number $z$ such that $x_1 < z < x_2$ and $f'(z) = 0$ ($f'$ is the derivative of $f$)

**Answer:** Let $f(x) = x^3 + 5x - 2$. Note that $f(x)$ is a differentiable function with $f'(x) = 3x^2 + 5$. Assume by contradiction that the equation $f(x) = 0$ has two real solutions, $x_1, x_2$, with $x_1 < x_2$. By Rolle's theorem, there must exist a real number $z$ ($x_1 < z < x_2$) such that $f'(z) = 0$. But $f'(z) = 3z^2 + 5 \geq 5 > 0$ cannot be zero, because the square of any real number is $\geq 0$. This is a contradiction.

**2. a.** Give the definition of the greatest common divisor of two natural numbers.

**Answer:** Let $a, b$ be two natural numbers. The greatest common divisor of $a, b$ is a natural number $d$ such that $d$ divides $a$, $d$ divides $b$, and if $c$ is any natural number such that $c$ divides $a$ and $c$ divides $b$, then $c$ divides $d$.

Formally: $d|a$ and $d|b$ and ($c|a$ and $c|b \Rightarrow c|d$)

**b.** Let $a, b$ be two arbitrary natural numbers, and let $d = \gcd(a, b)$. Prove that for every natural number $n$, $\gcd(na, nb) = n\gcd(a, b)$.

Let $d = \gcd(a, b)$. We need to prove that $nd$ satisfies the three requirements in the definition of gcd for $na, nb$:

- $nd|na$
- $nd|nb$
- if $c|na$ and $c|nb$, then $c|nd$.

We know that $d|a$, so we can write $a = kd$ with $k =$ integer. Then we have $na = nkd = k(nd)$, so $nd|na$. Similarly, we know that $d|b$, so we can write $b = ld$, with $l =$ integer. Then we have $nb = nld = l(nd)$, so $nd|nb$.

Now we prove the last part: Assume that $c$ is a natural number, and that $c|na$ and $c|nb$. We know that $d = \gcd(a, b)$ is a linear combination of $a$ and $b$, so we can write $d = ax + by$ with $x, y =$ integer. Then we also have $nd = nax + nby$. since $c$ divides $na$ and $nb$, it follows that $c$ also divides $nax$ and $nby$, and therefore $c$ divides $nax + nby = nd$. Since $c$ divides $nd$, it follows that $c \leq nd$.

**3.** For each of the following statements, decide if the statement is true or false and give a brief justification.

a. $\mathbf{N} \subseteq \mathbf{Q}$ **Answer:** true; every natural number is also a rational number b. $[1, 2] = \{1, 2\}$ **Answer:** false; there are other numbers in $[1, 2]$ other than 1 and 2; for example 1.5.

c. $(2, 3) \subseteq (1.5, 2.5)$ **Answer:** false; there are numbers in $(2, 3)$ that are not in $(1.5, 2.5)$; for example 2.8.

d. $\{x \in \mathbf{R} : x^2 + 2x + 1 = 0\} = \{-1\}$ **Answer:** True; $x = -1$ is the only solution of the equation $x^2 + 2x + 1 = 0 \Leftrightarrow (x + 1)^2 = 0$.

e. $\{x \in \mathbf{N} : 1 \leq x \leq 7\} \subseteq \{x \in \mathbf{N} : x^2 \leq 66\}$

**Answer:** true; for every number $x \leq 7$, we have $x^2 \leq 49 < 66$

**4.** (14 pts) Let $A, B$ be two arbitrary sets in a fixed universe $U$. Prove that $(A \cap B)^c = A^c \cup B^c$. (**note:** A picture might be helpful but it is not a proof. A proof should start from definitions.)

**Answer:**

We have $x \in (A \cap B)^c \Leftrightarrow x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B \Leftrightarrow x \in A^c$ or $x \in B^c \Leftrightarrow x \in A^c \cup B^c$.

**5.** (15 pts) Use the principle of mathematical induction (PMI) to prove that for all natural numbers $n$ we have

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

(the expression on the left hand side is the sum of all the natural numbers from 1 to $n$).

**Answer:** (i) check that the statement is true for $n = 1$:

LHS= 1; RHS= $\frac{1*2}{2} = 1$

(ii) check that $P(n) \Rightarrow P(n+1)$. Assume that $P(n)$ is true, so

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

To show that $P(n+1)$ is true, we calculate

$$1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

which is the desired formula.

The next two problems are "proofs to grade." **for full credit:** you need to analyze each statement in the above proof and decide if it is correct or not. Then you need to decide if the "proof" is indeed a proof of the "claim" or not. If the proof is incorrect you need to say precisely what is incorrect about it **and what should be done to correct it.**(for instance if the **claim** is incorrect, then you should give a counterexample; if the proof is incomplete, you should complete it).

**6. Claim:** Let $A, B, C$ be arbitrary sets. If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, then $A \cap C \neq \emptyset$.

**"Proof":** Assume $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$. Then there exists an element $x$ such that $x \in A \cap B$. By the definition of $A \cap B$, $x \in A \cap B \Rightarrow x \in A$. Similarly, since $B \cap C \neq \emptyset$, there exists an element $x$ such that $x \in B \cap C$. By the definition of $B \cap C$, $x \in B \cap C \Rightarrow x \in C$. Since we have $x \in A$ and $x \in C$, it follows that $x \in A \cap C$, and thus $A \cap C \neq \emptyset$.

**Answer:** The proof is incorrect because it uses the same name $x$ for two elements that could potentially be different (there is no reason to assume that the element that $A$ and $B$ have in common is the same as the element that $B$ and $C$ have in common).

In order to correct the problem, we give a counterexample to show that the claim is actually false: let $A = \{1, 2, B = \{2, 3\}, C = \{3, 4\}$. Then we have $A \cap B = \{2\} \neq \emptyset, B \cap C = \{3\} \neq \emptyset$, but $A \cap C = \emptyset$.

**7.** "Proof to grade:"

**Claim:** For all natural numbers $n \in \mathbf{N}$, $n^3 + 44n$ is divisible by 3.

**"Proof:"** We do a proof by induction. We need to check that conditions (i) and (ii) in the statement of the PMI are satisfied.

(i) If $n = 1$: $1^3 + 44 * 1 = 45$, which is divisible by 3.

(ii) Let $n \in \mathbf{N}$ be an arbitrary natural number. Assume that the statement is true for $n$. Then $n^3 + 44n$ is divisible by 3. Therefore $(n+1)^3 + 44(n+1)$ is divisible by 3.

The proof is complete by the PMI (principle of mathematical induction).

**Answer:** The proof follows the correct outline for a proof by induction. However, the proof is incomplete because there is no proof given for the conclusion "Therefore $(n+1)^3 + 44(n+1)$ is divisible by 3" in part (i).

To correct the problem:

We are assuming that $n^3+44n$ is divisible by 3, so we can write $n^3+44n = 3k$ for some integer $k$. Then we have $(n+1)^3+44(n+1) = n^3+3n^2+3n+1+44n+44 = (n^3+44n)+(3n^2+3n+45) = 3k + 3(n^2+n+15) = 3(k+n^2+n+15)$, which is $3*$ an integer, and therefore it is divisible by 3.