Math 300 Fall 2013 Exam 2

1. Consider the statement: "For every integer x there is an integer y such that $x+y$ is an odd integer."

a. Write the above statement symbolically using quantifiers.

Answer: $(\forall x)(\exists y)(x+y$ is odd$)$ (universe= all integers)

b. Give a proof of the statement.

Answer: Let x be an arbitrary integer, and let $y = x+1$. Then $x+y = 2x+1$ is odd, since it has the correct form for an odd integer ($2k+1$, with $k = x$, an integer).

2. Prove that $\sqrt{2}$ is not a rational number. Give a proof by contradiction.

Answer: Assume by contradiction that $\sqrt{2}$ is a rational number, so we can write $\sqrt{2} = p/q$ with p, q integers. Moreover, we may assume that p and q don't have any common divisors (because if they do have a common divisor we can simplify the fraction and replace p, q by the numerator and denominator of the simplified fraction).

We get $2 = p^2/q^2$, so $2q^2 = p^2$, which show that p^2 is an even integer. This implies that p must be an even integer (because otherwise p is odd, so $p = 2k+1 \Rightarrow p^2 = 4k^2+4k+1$, which would be odd, and this contradicts what we know – p^2 is even). So we may write $p = 2k$ for some integer k. Then $p^2 = 4k^2$, and plugging this into the previous equation we get $2q^2 = 4k^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even. So we may write $q = 2l$ for some integer l. But now p, q are both multiplies of 2, which contradicts our assumption that p, q don't have any common divisors. This is a contradicition.

3. Let x, y, z be integers. Give a proof by contrapositive of the statement:

"If x does not divide yz, then x does not divide z."

Answer: The contrapositive is: "If x divides z, then x divides yz." Assume that x divides z, so we have $z = xk$ for some integer k. Then $yz = yxk = x(yk) = yl$ where $l = yk$ is an integer, so x divides yz.

4. Prove that the line of equation $x+y=2$ and the circle of equation $x^2+y^2=1$ do not intersect. Give a proof by contradiction.

Answer: Assume by contradiction that the circle and the line have a point in common, say that (a, b) is the point of intersection. Then we have $a+b=2$ and $a^2+b^2=1$. Plug in $b = 2-a$ from the first equation into the second equation. We get $a^2+(2-a)^2 = 1$, so $a^2+4-4a+a^2 = 1 \Rightarrow 2a^2-4a+3 = 0 \Rightarrow 2(a^2-2a+1)+1 = 0 \Rightarrow 2(a-1)^2 = -1$. This is not possible, because the square of any real number is always ≥ 0.

5. For each of the following statements, decide if the statement is true or false. Then, give a proof (if true) or a counterexample (if false):

a. For all positive real numbers x, $x^2 - x > 0$.

Answer: False. $x = 1/2$ is a counterexample since $1/4 - 1/2 = -1/2 < 0$.

b. For every positive real number x, there exists a positive real number y such that $xy - y = 1$.

Answer: False. Take $x = 1$. Then $xy - y = y - y = 0$ no matter what value of y is used, so it cannot be equal to 1.

c. For every integer t, there exist integers n, m such that $2n+3m = t$ (hint: it might help to first consider the case when $t = 1$).

Answer: True. Let t be a fixed integer. Take $n = -t$ and $m = t$. Then $2n+3m = -2t+3t = t$.

6. This is a "proof to grade" question. Consider the statement:

"Claim:" For all real numbers t, if t is irrational then $t+6$ is also irrational.

"Proof": Fix a real number t. Assume that t is rational, so we can write $t = \dfrac{p}{q}$ with p, q integers, $q \neq 0$. Then $t + 6 = \dfrac{p}{q} + 6 = \dfrac{p+6q}{q}$. Since $p + 6q$ is also an integer, it follows that $t + 6$ is a rational number.

for full credit: you need to analyze each statement in the above proof and decide if it is correct or not. then you need to decide if the "proof" is indeed a proof of the "claim" or not. If the proof is incorrect you need to say precisely what is incorrect about it and what should be done to correct it.

Answer: The proof is correct, but it is NOT a proof of the "Claim"; it is a proof of $\sim p \Rightarrow \sim q$ (which is equivalent to the converse $q \Rightarrow p$) instead of a proof of $p \Rightarrow q$.

7. This is a "proof to grade" question. Consider the statement:

"Claim:" There do not exist integers n, m such that $n^2 = 4m + 3$.

"Proof:" Assume by contradiction that there exist integers n, m with $n^2 = 4m + 3$. Since n^2 is odd, we know that n must be odd also (previously known fact). Say that $n = 2k + 1$ where k is an integer. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. It follows that $4m + 3 = 4k(k+1) + 1$, so $2 = 4k(k+1) - 4m$, and therefore 2 is divisible by 4. This is a contradiction.

for full credit: you need to analyze each statement in the above proof and decide if it is correct or not. then you need to decide if the "proof" is indeed a proof of the "claim" or not. If the proof is incorrect you need to say precisely what is incorrect about it and what should be done to correct it.

Answer: The proof is correct.