Math 300 Fall 2013 Exam 1

1. Give the truth value of each of the following propositions. Give a brief justification for each answer.

a. Either $3+\pi$ is a rational number or $4>0$.

Answer: This has the form $p \vee q$ where p is F and q is T, so $p \vee q$ is T.

b. It is not the case that 20 is a prime number, and it is not the case that 7 is a prime number.

Answer: This has the form $\sim p \wedge \sim q$, where $p =$ "20 is a prime number" is F, so $\sim p$ is T, and $q =$ "7 is a prime number" is T, so $\sim p$ is F. Therefore $\sim p \wedge \sim q$ is F.

c. Both 15 and 12 are odd numbers.

Answer: This has the form $p \wedge q$ where p is T and q is F so $p \wedge q$ is F.

d. If 15 is an odd number, then 12 is an odd number.

Answer: This has the form $p \Rightarrow q$; p is T and q is F, therefore $p \Rightarrow q$ is F.

e. 15 is an odd number, if 12 is an odd number.

Answer: This has the form $q \Rightarrow p$. $q =$ "12 is an odd number" is F and $p =$ "15 is an odd number" is T, so $q \Rightarrow p$ is T.

f. $3+\pi$ is a rational number if and only if $3-\pi$ is a rational number. **Answer:** This has the form $p \Leftrightarrow q$. p is F and q is F, therefore $p \Leftrightarrow q$ is T.

g. $1+1=2$ is sufficient for $1+1=3$.

Answer: This has the form $p \Rightarrow q$. p: "$1+1=2$" is T; q: $1+1=3$ is F. Therefore $p \Rightarrow q$ is F.

h. $1+1=2$ is necessary for $1+1=3$.

Answer: This has the form $q \Rightarrow p$. q: "$1+1=3$" is F and p : "$1+1=2$" is T. Thereofore $q \Rightarrow p$ is T.

2. a. Make a truth table for the propositional form $(P \wedge Q) \vee (\sim P \wedge \sim Q)$

Answer:

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	T	T

b. Assume that $(P \wedge Q) \vee (\sim P \wedge \sim Q)$ is true, and that P is false. What can you say about Q?

Answer: The only row of the truth table where both of the assumptions are satisfied is the fourth row. Therefore, Q must be F.

3. Show that $(P \vee Q) \Rightarrow R$ is equivalent to $\sim R \Rightarrow (\sim P \wedge \sim Q)$.

note: the version that you had on the exam had a typo that made the two propositional forms not equivalent to each other. this is the version that I meant to ask instead.

Answer: We know that the given implication is equivalent to its counterpositive, which is $\sim R \Rightarrow \sim (P \vee Q)$. By the DeMorgan Laws, we also know that $\sim (P \vee Q)$ is equivalent to $\sim P \wedge \sim Q$, so now the given implication is equivalent to $\sim R \Rightarrow (\sim P \wedge \sim Q)$.

4. Which of the following are denials of $P \Rightarrow (Q \wedge R)$? For each of the choices, explain briefly why it is or it is not a denial of the original statement:

Answer: A denial of an implication $P \Rightarrow Q$ is $P \wedge \sim Q$ (since the only time when the implication is false is when P is true and Q is false).

In the case of the implication $P \Rightarrow (Q \wedge R)$, this takes the form $P \wedge \sim (Q \wedge R)$. Using the De Morgan Laws, this is equivalent to $P(\sim Q \vee \sim R)$, so **d.** is a denial of the original implication. In order to justify why each of the other choices is not a denial, we will show one choice of truth values for P, Q, R that make the original implication, and the statement we are considering have the same truth values (therefore the denial of the original implication and the statement we are considering are not equivalent).

a. $P \Rightarrow (\sim Q \vee \sim R)$

Answer: If P is false, then the original implication and this statement are both true. Therefore, this is not a denial.

b. $P \Rightarrow (\sim Q \wedge \sim R)$

Answer: If P is false, then the original implication and this statement are both true. Therefore, this is not a denial.

c. $P \wedge \sim Q \wedge \sim R$

Answer: If P, Q are T and R is F, then the original implication is F (since P is T and $Q \wedge R$ is F), and this statement is also F (since $\sim Q$ is F, making the whole conjunction F).

d. $P \wedge (\sim Q \vee \sim R)$.

Answer: This is a denial. See reason above.

5. Give the truth value of each of the following propositions (the universe is specified for each part). Give a brief justification for each answer.

a. $(\forall x)(x^2 \geq 1 \Leftrightarrow x \geq 1)$ (universe: all real numbers)

Answer: False. This has the form $(\forall x)(p(x) \Leftrightarrow q(x))$. However, for $x < -1$, $p(x)$ is T and $q(x)$ is F, therefore we found (at least) one choice of x for which $p(x) \Leftrightarrow q(x)$ is F.

b. $(\exists x)(x \text{ is odd } \wedge\ x + 1 \text{ is odd})$ (universe: all integers)

Answer: False. If x is odd then $x + 1$ must be even.

c. $(\exists x)(x \text{ is odd } \Rightarrow\ x + 1 \text{ is odd})$ (universe: all integers)

Answer: True. This has the form $(\exists x)(p(x) \Rightarrow q(x))$. For this type of statement to be true, it is enough to find an example of x that makes the implication true. For $x = 4$, $p(x)$ is F and $q(x)$ is T, therefore $p(x) \Rightarrow q(x)$ is T.

d. $(\forall x)(x \geq 0 \Rightarrow (\exists! y)(x = y^2))$ (universe: all real numbers)

Answer: False. For $x > 0$, we can take $y = \sqrt{x}$ and also $y = -\sqrt{x}$. Both choices of y make the equation $x = y^2$ true, therefore the uniqueness part of the statement is false.

e. $(\forall x)((\exists y)(x = y^2) \Leftrightarrow x \geq 0)$ (universe: all real numbers)

Answer: True. This has the form $(\forall x)(p(x) \Leftrightarrow q(x))$. To prove this: Let x be a fixed arbitrary real number. We will show $p(x) \Rightarrow q(x)$, and $q(x) \Rightarrow p(x)$.

Show $p(x) \Rightarrow q(x)$: Assume that $x = y^2$ for some real number y. Then $x \geq 0$ because the square of any real number is ≥ 0.

Show $q(x) \Rightarrow p(x)$. Assume that $x \geq 0$. Take $y = \sqrt{x}$. Then we have $x = y^2$.

6. Give symbolic translations of each of the following statements. Use appropriate quantifiers for each of the variables x, y.

a. For every real number x, there exists a real number y such that $x \leq y$.

Answer: $(\forall x)(\exists y)(x \leq y)$.

b. There exists a real number x such that $x \leq y$ for every real number y.

Answer: $(\exists x)(\forall y)(x \leq y)$.

7. Give a proof from scratch (using just definitions, not any other previously known facts) of the following statement:

(a) If x is an integer, then $x(x+3)$ is an even integer.

Answer: We consider two possible cases: x is even or x is odd.

Case 1: Assume x is even, so $x = 2k$ where k is an integer. Then $x(x+3) = 2k(2k+3) = 2(k(2k+3)) = 2l$, where $l = k(2k+3)$ is an integer.

Case 2: Assume x is odd, so $x = 2k+1$, where k is an integer. Then $x(x+3) = (2k+1)(2k+4) = 2(2k+1)(k+2) = 2l$, where $l = (2k+1)(k+2)$ is an integer.

(b) Let x, y, z be integers. If x divides y and y divides z, then x divides z.

Answer: Assume that x divides y and y divides z. Therefore we have $y = kx$ and $z = ly$ with $k, l =$integers. Then we have $z = ly = l(kx) = (lk)x$ and lk is an integer.

8. Prove the following:

Let x be a real number. If $x^3 - 2x^2 < 0$, then $2x + 7 < 11$

(you may use the technique of working backwards from the desired conclusion)

Answer: The desired conclusion is $2x + 7 < 11 \Leftrightarrow 2x < 4 \Leftrightarrow x < 2$. Thus it will be enough to show that $x < 2$, since this implies that $2x + 7 < 11$ (**note the use of biconditionals!**).

Now assume that $x^3 - 2x^2 < 0$. This can be written as $x^2(x-2) < 0$. Since x is a real number, we know that $x^2 \geq 0$. The only way that the product can be negative is if one factor is positive and the other is negative. Thus we must have $x - 2 < 0 \Leftrightarrow x < 2$. As noted above,this implies the desired conclusion.