Math 172 Spring 2012 Handout 1: Discrete and continuous models; Linear and exponential growth

Introduction and terminology

We will model the size of a population, P, as a function of time, t. Thus, $P = P(t)$. Sometimes we might use N instead of P, or n instead of t.

In a **discrete model** the time t is measured at discrete intervals, e.g. year 1, year 2, etc., or day 1, day 2, etc., depending on what units are being used. In this case the discrete variable t only takes on non-negative integer values, $t = 0, 1, 2, etc.$. Sometimes we use n instead of t in a discrete model.

We are concerned with the **change** in the size of a population. In a discrete model, this is denoted ΔP.

$$\Delta P = P(n+1) - P(n).$$

An equation that gives information about ΔP is called a **difference equation**, or a **dynamic model**. When we are able to, we will use the difference equation in order to find an explicit formula for $P(t)$. This explicit formula is called the **explicit solution** to the difference equation (also called a **static model**).

We can always translate the difference equation into a **recursive equation**, which tells us how to find $P(n+1)$ once $P(n)$ is known:

$$P(n+1) = P(n) + \Delta P.$$

Examples

Discrete linear model: ΔP is constant.

For example $\Delta P = 3$. This means that for each time unit that passes, the size of the population increases by 3 individuals. In order to be able to find the explicit solution, $P = P(t)$, one also needs to know the **initial value** of the population, P_0. Let's assume $P_0 = 100$, so the population starts out with 100 individuals, and increases by 3 individuals per unit of time. Let's say that time is measured in days.

We want to find the explicit solution, that is: how many individuals are there after t days? We have:

$$P(0) = 100, P(1) = 100 + 3 = 103, P(2) = 103 + 3 = 106, \text{etc}$$

In general, after t years, $3 \times t$ individuals will be added to the initial population of 100, so $P(t) = 100 + 3t$ is the explicit solution. Note that the general solution is given by a **linear function**; if we graph P as a function of t, the graph will be a line with slope equal to 3 and vertical intercept equal to 100.

Discrete exponential model: ΔP is proportional to the value of P. This is usually the case when the change in the population is due to unrestricted reproduction, that is: each reproductive pair gives rise to a fixed number of offspring, let's say 6 (for example) per year.

Let's make the simplifying assumption that the entire population consists of reproductive adults (and the new offspring become reproductive adults as soon as they are born - this is of course not realistic - we will later introduce the concept of populations with age structure in order to make this model more realistic).

Thus, if the total size of the population is P, we have $\frac{1}{2}P$ pairs of reproductive adults, which give rise to $\frac{1}{2}P \times 6 = 3P$ new offspring per year.

The difference equation in this example is:

$$\Delta P = 3P.$$

The recursive equation is

$$P(t+1) = P(t) + 3P(t).$$

Note that as the size of P increases, so does the value of ΔP. Thus, this population will increase faster and faster. This is a trademark of the exponential function.

Let us find the explicit solution corresponding to this difference equation. Again, let's assume that the initial population is $P_0 = 100$.

For the first year, we have $\Delta P = 300$, so $P(1) = P(0) + \Delta P = 100 + 300 = 400$. For the second year, $\Delta P = 3 \times 400 = 1200$, so $P(2) = P(1) + 1200 = 1600$, etc.

In order to write down the general formula, note that the recursive equation (see above) can be written as

$$P(t+1) = 4P(t).$$

Thus, the size of the population gets multiplied by 4 with each passing year. After t years, the size of the population gets multiplied by 4 t times, that is, it gets multiplied by 4^t, giving us the general formula:

$$P(t) = P_0 \times 4^t = 100 \times 4^t$$

As expected, the general formula is given by an **exponential function** (the presence of the variable t in the exponent indicates that this is an exponential function).

(please note: 100×4^t is not the same as 400^t)

In general, the difference equation for the discrete exponential model takes the form

$$\Delta P = rP \tag{1}$$

and the general solution is:

$$P(t) = P_0(1+r)^t.$$

where P_0 denotes the initial value of the population ($P_0 = P(0)$).

Terminology and units: We will refer to the amount of change ΔP as the **net growth rate** and the constant r in the difference equation 1 above as the **per capita growth rate**. This terminology will be in effect whenever we discuss an exponential model.

Regarding units: the size of the population $P(t)$ is usually given in individuals, although sometimes it might be given in thousands of individuals, or millions of individuals, etc. In case of population consisting of microscopic individuals (for instance bacteria) we might measure the size of the population in terms of its mass (we weigh the population rather than counting the individuals) and the units might be grams or kg.

The net growth rate ΔP is measured in individuals per unit of time, or whatever units are used for $P(t)$ per unit of time. For instance if $P(t)$ is measured in thousands of individuals, and time t is measured in years, then ΔP will be given in terms of thousands of individuals per year.

The per capita growth rate (r in equation 1) is usually given as individuals per individual per year. Note that $r = \Delta P/P$, so the units of r are units of ΔP, divided by units of P.

The per capita growth rate might also be given as a **percentage** per unit of time. For example, we might be told that the population grows at a rate of 12% per year. This means that $r = 12/100$, so the corresponding difference equation is $\Delta P = 0.12P$.

We might also encounter situations when a population is decreasing rather than increasing. This translates into a negative growth rate.

Continous models

In a continuous model, the time t is a continuous variable. That is, we keep track of what happens to the population all the time instead of just at discrete intervals (such as once a day, or once a year, as we would in a discrete model).

We use **differential equations** to study continuous models. A differential equation gives information about the **derivative** $\frac{dP}{dt}$ of the population function.

The derivative $\frac{dP}{dt}$ is defined as the limit of $\frac{\Delta P}{\Delta t}$ when Δt becomes smaller and smaller. Equivalently, we may think of the derivative as the value of ΔP when the unit for time is chosen to be equal to Δt (so that $\Delta t = 1$), and this unit is considered to be very small. For this reason, the derivative is also referred to as **instantaneous rate of change**.

The exponential continous model

The exponential continous model assumes that the instantanous rate of change is proportional to the size of the population. We have a differential equation:

$$\frac{dP}{dt} = rP,$$

where r is a constant.

Recall from calculus that the solution of this differential equation is given by

$$P = P_0 e^{rt}$$

We can check this by plugging the function $P = P_0 e^{rt}$ into the differential equation by taking the derivative. Recall that the derivative of e^{rt} is re^{rt}. Since P_0 is a multiplicative constant, it stays the same when taking the derivative. Thus

$$\frac{d(P_0 e^{rt})}{dt} = P_0 r e^{rt} = rP,$$

as required.

Note: Please note that even though both the discrete model $\Delta P = rP$ and the continuous model $\frac{dP}{dt} = rP$ assume the same per capita growth rate r, the formulas for the general solutions are different.

We now explore the relationship between the continous and the discrete model.

The continuous model is equivalent to a discrete model in which the time step Δt is allowed to get smaller and smaller, while in the original discrete model we fix the time step to be equal to 1 unit of time. Let's consider various discrete models in which the time step is $\Delta t = \frac{1}{n}$ (getting smaller and smaller when n gets larger).

This correspongs to a per capita growth rate equal to $\frac{r}{n}$ (if the population grows by rP in one unit of time, it will only grow by $(rP)/n$ in $1/n$ of a unit of time).

The equation $\Delta P = \frac{r}{n} P$ gives rise to $P = P_0(1 + \frac{r}{n})^t$, where t is now measured in $1/n$ units of time. We go back and measure t in the original units. Note that 1 original unit= n of the smaller units, thus t original units = nt smaller units. When the time t is expressed in the original units, the formula becomes

$$P = P_0(1 + \frac{r}{n})^n t$$

Since $e^r = \lim_{n \to \infty}(1 + \frac{r}{n})^n$, the limit of the solutions for the discrete models with $n \to \infty$ is indeed the solution for the continous model, $P = P_0 e^{rt}$.