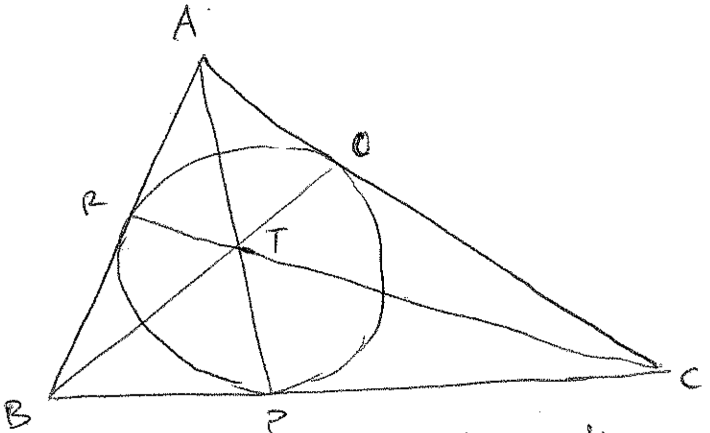
4A.1.



If T is the <u>incenter</u> then AP is an angle bisector of $\angle BAC$, but also $AP \perp BC$, so $\triangle APB \cong \triangle APC$ so $AB = AC$. Similarly $AB = BC$.

If T is the circumcenter then $BP = PC$, $CQ = QA$, $AR = RB$. But we also know ~~[illegible]~~ $BR = BP$ (because $\triangle BTR \cong \triangle BTP$), $RA = AQ$, $QC = CP$, so these are all equal so $AB = BC = CA$.

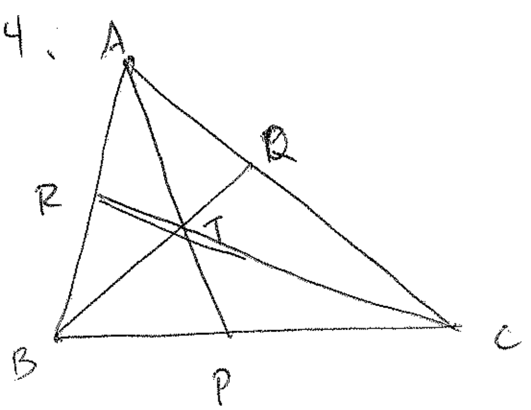
If T is the centroid, same argument works.

If T is the orthocenter, then $AP \perp BC$.

If T is

But if I is the incenter, $IP \perp BC$ also, so T is on the angle bisector again and we argue as before.

4A.4.



$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$

So
$$\frac{CQ}{QA} = \left(\frac{AR}{RB} \cdot \frac{BP}{PC}\right)^{-1} = \left(\frac{2}{1} \cdot \frac{2}{1}\right)^{-1} = \frac{1}{4}.$$

So $\frac{CQ}{CA} = \frac{1}{5}$.

For $\frac{AT}{AP}$, see Thm. 4.4.
(I apologize, this was a little unfair. will count as bonus.)

1.

Say $AB = BC = CD = 1$

$$cr(A, B, C, D) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{2 \cdot 2}{3 \cdot 1} = \frac{4}{3}.$$

2. $$cr(B, A, C, D) = \frac{BC \cdot AD}{BD \cdot AC} = \frac{1 \cdot 3}{2 \cdot 2} = \frac{3}{4}.$$

$$cr(D, C, B, A) = \frac{DB \cdot CA}{DA \cdot CB} = \frac{2 \cdot 2}{3 \cdot 1} = \frac{4}{3}.$$

3. In general note $cr(A, B, C, D) = cr(D, C, B, A)$

because $$\frac{AC \cdot BD}{AD \cdot BC} = \frac{DB \cdot CA}{DA \cdot CB}$$

So we can flip backwards

Also $$cr(A, B, C, D) = cr(C, D, A, B)$$

because $$\frac{AC \cdot BD}{AD \cdot BC} = \frac{CA \cdot DB}{CB \cdot DA}.$$

Also we can combine these,

$$cr(A, B, C, D) = cr(B, A, D, C).$$

So it is enough to compute the cross ratios with A coming first. (Can rearrange any cross ratio as above to put A first and have the same value.)

$$cr(A, B, C, D) = \frac{2 \cdot 2}{3 \cdot 1} = \frac{4}{3}.$$

$$cr(A, C, B, D) = \frac{1 \cdot 1}{3 \cdot 1} = \frac{1}{3}.$$

$$cr(A, B, D, C) = \frac{3 \cdot 1}{2 \cdot 2} = \frac{3}{4}.$$

$$cr(A, C, D, B) = \frac{3 \cdot 1}{1 \cdot 1} = 3.$$

$$cr(A, D, B, C) = \frac{1 \cdot 1}{2 \cdot 2} = \frac{1}{4}.$$

$$cr(A, D, C, B) = \frac{2 \cdot 2}{1 \cdot 1} = 4.$$

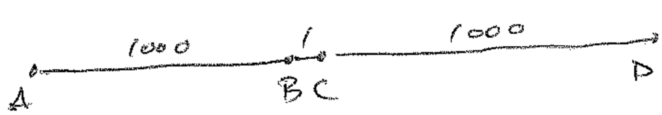
In Fancy Math 546 language:

The group Sym(4) acts on the set $\{\frac{1}{3}, 3, \frac{1}{4}, 4, \frac{4}{3}, \frac{3}{4}\}$

The stabilizers have size 4.

So the number of different values is $\frac{24}{4} = 6$.

4.

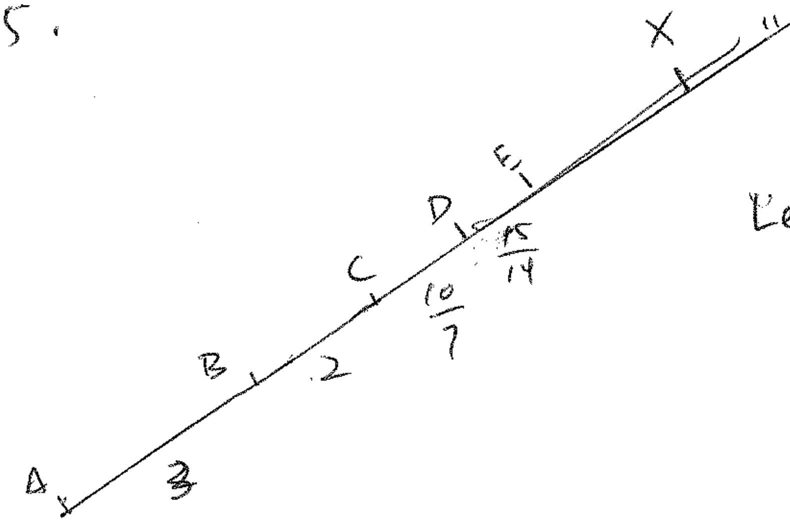


$$cr(A, B, C, D) = \frac{1001 \cdot 1001}{2001 \cdot 1}$$ which is close to $\frac{1000}{2}$.

The above shows how to get a small number:

$$cr(A, B, D, C) = \frac{2001 \cdot 1}{1001 \cdot 1001}$$ (inverse of above)

5.



"Evenly spaced" cross ratio is $\frac{4}{3}$.

Let $CD = c$.

$$\frac{5\cdot(2+c)}{(5+c)\cdot 2} = \frac{4}{3}$$

$$30 + 15c = 40 + 8c$$

$$10 = 7c, \text{ so } c = \frac{10}{7}.$$

Let $DE = d$.

$$\frac{\frac{24}{7}\left(\frac{10}{7} + d\right)}{\left(\frac{24}{7} + d\right)\cdot\frac{10}{7}} = \frac{4}{3}.$$

$$3(240 + 7\cdot 24\, d) = 4(240 + 70d)$$

$$720 + 504d = 960 + 280d$$

$$224d = 240.$$

$$d = \frac{240}{224} = \frac{60}{56} = \frac{15}{14}.$$

The next two are messy.
No way around that.

Horizon: $cr(A, B, C, X)$

as if $A'B' = B'C' = 1$, $C'X' = \infty$.

$$cr(A', B', C', X') = \frac{A'C'\cdot B'X'}{A'X'\cdot B'C'} = \frac{2\cdot(\infty+1)}{(\infty+2)\cdot 1} = \frac{2}{1}\cdot\frac{(\infty+1)}{(\infty+2)}$$

$$= \frac{2}{1}\cdot\cancel{\frac{\infty}{\infty}}$$

$$= 2.$$

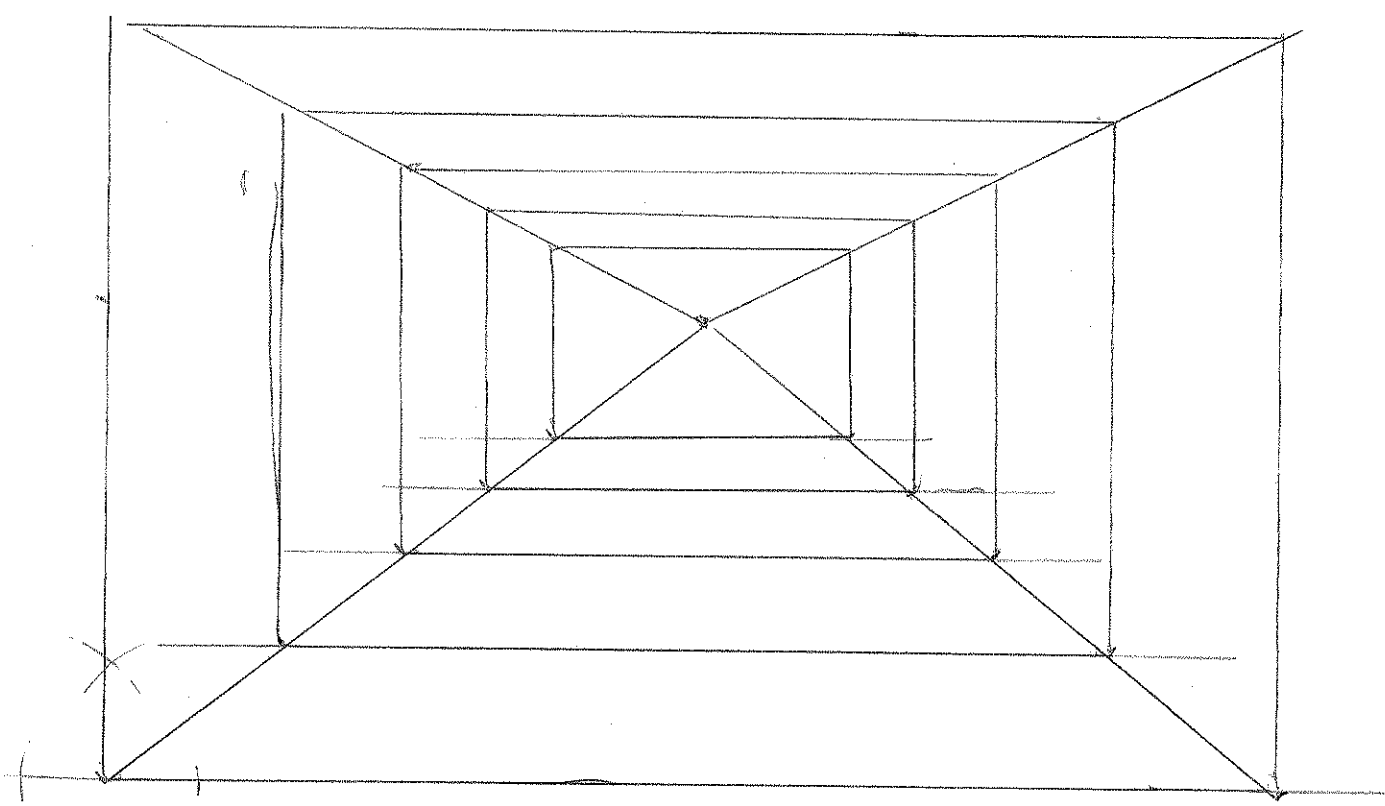
I am a trained professional. Don't try at home.

So, $2 = \frac{AC \cdot BX}{AX \cdot BC} = \frac{5 \cdot (2+x)}{(5+x) \cdot 2}$

$$4(5+x) = 5(2+x)$$
$$20 + 4x = 10 + 5x.$$

So $\underline{x = 10}$.

6. Step 1. Reproduce the lengths from before. (in bottom left)
   2. Do also in bottom right.
   3. Connect dots at bottom.
      These lines are guaranteed to be parallel to each other.
   4. Draw perpendiculars. (Your compass set has a right angle, so "cheat" if you prefer.)
   5. Draw a line from center to top left to meet the perpendiculars.
   6. Draw perpendiculars to top right from left + bottom.



"Use the force, Luke."