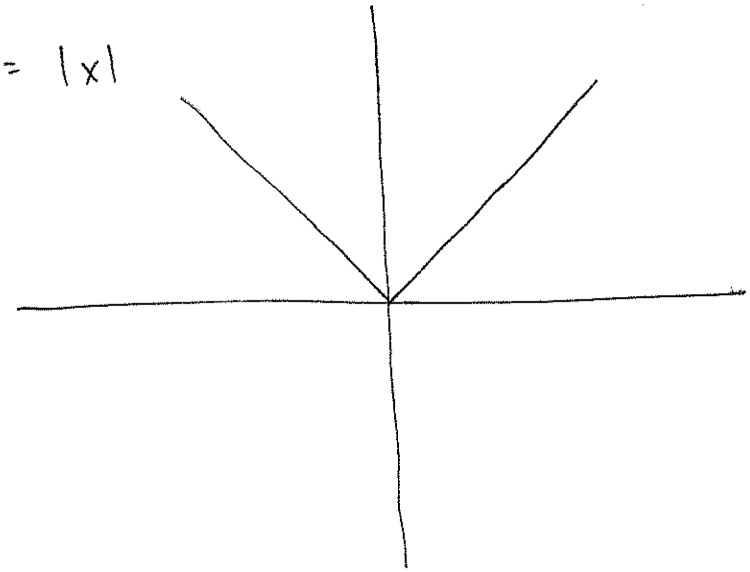
Exam 2 solutions.

1. (12 pts.) $f(x) = |x|$



This is not differentiable at $x = 0$ because it has a corner (or a "cusp"). There is no way to define a slope at that point because the graph changes direction.

2. (13 pts.)

$$g'(t) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t}\sqrt{t+h}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h \cdot \sqrt{t}\sqrt{t+h}}$$

$$= \lim_{h \to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h \cdot \sqrt{t}\sqrt{t+h}} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}$$

$$= \lim_{h \to 0} \frac{t - (t+h)}{h \cdot \sqrt{t}\sqrt{t+h}\left(\sqrt{t} + \sqrt{t+h}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{t}\sqrt{t+h}\left(\sqrt{t} + \sqrt{t+h}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{t}\sqrt{t+h}\left(\sqrt{t} + \sqrt{t+h}\right)} = \frac{-1}{\sqrt{t} \cdot \sqrt{t}\left(\sqrt{t} + \sqrt{t}\right)}$$

$$= -1/2t^{3/2}.$$

3. $f'(x) = 100x^{99}$

(12 pts.) $f''(x) = 100 \cdot 99 \, x^{98}$,

and so on down to

$f^{(100)}(x) = 100 \cdot 99 \cdot 98 \cdot \cdots \cdot 1 \cdot x^0$ by the Power Rule.

$x^0 = 1$ so this is a constant, so

$f^{(101)}(x) = 0$. If you take derivatives of 0, you keep getting 0, so $f^{(500)}(x) = 0$.

4. $\frac{d}{d\theta} \csc\theta = -\csc\theta\cot\theta$, $\frac{d}{d\theta}\cot\theta = -\csc^2\theta$.

(13 pts.)

So, $\frac{d}{d\theta}(\csc\theta + e^\theta \cot\theta)$

$= -\csc\theta\cot\theta + e^\theta \frac{d}{d\theta}(\cot\theta) + \frac{d}{d\theta}(e^\theta) \cdot \cot\theta$

$= -\csc\theta\cot\theta - \csc^2\theta \, e^\theta + e^\theta \cdot \cot\theta$.

5. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, so

(13 pts.)

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}\left((2x^2 + 2y^2 - x)\right)^2$

$2x + 2y\frac{dy}{dx} = 2 \cdot (2x^2 + 2y^2 - x)\frac{d}{dx}(2x^2 + 2y^2 - x)$

$= 2 \cdot (2x^2 + 2y^2 - x)\left(4x + 4y\frac{dy}{dx} - 1\right)$.

(cont.)

Solution 1. (direct way)

Solve for $\frac{dy}{dx}$:

$$2x + 2y\frac{dy}{dx} = 2\Big(8x^3 + 8x^2y\frac{dy}{dx} - 2x^2 + 8xy^2 + 8y^3\frac{dy}{dx} - 2y^2 - 4x^2 - 4xy\frac{dy}{dx} + x\Big)$$

$$x + y\frac{dy}{dx} = 8x^3 + 8x^2y\frac{dy}{dx} - 6x^2 + 8xy^2 + 8y^3\frac{dy}{dx} - 2y^2 - 4xy\frac{dy}{dx} + x$$

$$y\frac{dy}{dx} - 8x^2y\frac{dy}{dx} - 8y^3\frac{dy}{dx} + 4xy\frac{dy}{dx} = 8x^3 - 6x^2 + 8xy^2 - 2y^2 + x - x$$

$$\frac{dy}{dx}\left(y - 8x^2y - 8y^3 + 4xy\right) = 8x^3 - 6x^2 + 8xy^2 - 2y^2$$

$$\frac{dy}{dx} = \frac{8x^3 - 6x^2 + 8xy^2 - 2y^2}{y - 8x^2y - 8y^3 + 4xy}.$$

Plug in $x = 0$, $y = 1/2$:

$$\frac{dy}{dx} = \frac{8\cdot 0^3 - 6\cdot 0^2 + 8\cdot 0\cdot\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2}{\frac{-1}{2} - 8\cdot 0^2\left(\frac{1}{2}\right) - 8\cdot\left(\frac{1}{2}\right)^3 + 4\cdot 0\cdot\left(\frac{1}{2}\right)} = \frac{-1}{-\frac{1}{2} - \frac{1}{2}}$$

$$= \frac{-1}{-1} = 1.$$

So, the tangent line is $y - \frac{1}{2} = 1\cdot(x - 0)$ or $y = x + \frac{1}{2}$.

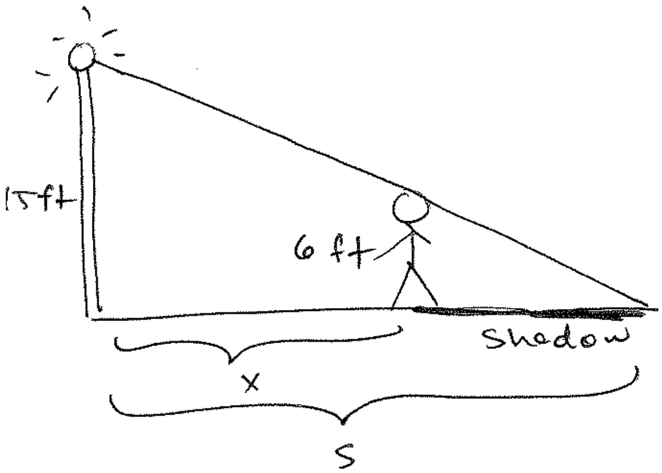
Solution 2. (shortcut) Plug in $x = 0$, $y = 1/2$ to last equation on previous page:

$$2\cdot 0 + 2\cdot\frac{1}{2}\cdot\frac{dy}{dx} = 2\cdot\left(2\cdot 0^2 + 2\cdot\left(\tfrac{1}{2}\right)^2 - 0\right)\left(4\cdot 0 + 4\cdot\left(\tfrac{1}{2}\right)\frac{dy}{dx} - 1\right)$$

$$\frac{dy}{dx} = 2\cdot\frac{1}{2}\cdot\left(2\frac{dy}{dx} - 1\right) = 2\frac{dy}{dx} - 1$$

so $\frac{dy}{dx} = 1$. Now find the tangent line in the same way.

6.



Let

t = time in seconds

x = distance from woman to pole

s = distance from tip of shadow to base of pole.

Want : $\frac{ds}{dt}$.

Know : $x = 40$ ft (now)

$\frac{dx}{dt} = 5 \frac{ft}{s}$ always.

By similar triangles, $\frac{s-x}{6} = \frac{s}{15}$

$$\text{So} \quad \frac{s}{6} - \frac{x}{6} = \frac{s}{15}$$

$$s \cdot \left(\frac{1}{6} - \frac{1}{15}\right) = \frac{x}{6}$$

$$s \cdot \left(\frac{5}{30} - \frac{2}{30}\right) = \frac{x}{6}$$

$$s \cdot \frac{3}{30} = \frac{x}{6}$$

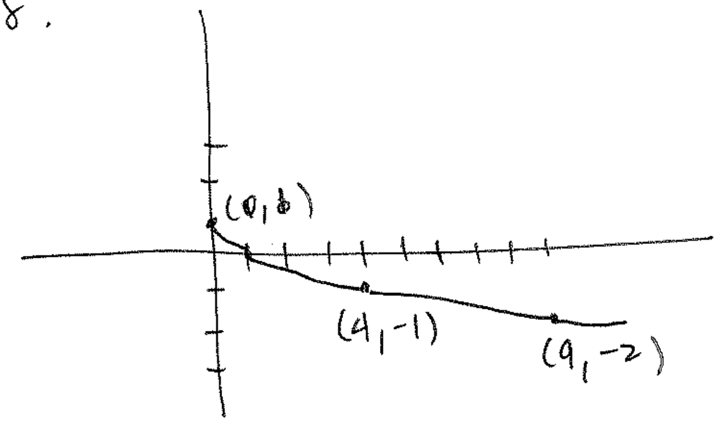
$$s = x \cdot \frac{1}{6} \cdot \frac{30}{3} = \frac{5}{3} x.$$

So $\frac{ds}{dt} = \frac{5}{3} \frac{dx}{dt}$.

So when $\frac{dx}{dt} = 5 \frac{ft}{s}$, $\frac{ds}{dt} = \frac{5}{3} \cdot 5 \frac{ft}{s} = \frac{25}{3} \frac{ft}{s}$.

7. $f'(x) = \dfrac{1}{\ln(1+2x)} \cdot \dfrac{d}{dx}(1+2x) = \dfrac{2}{\ln(1+2x)}$.

8.



We can see from the graph that the function is ~~also~~ always decreasing. So there is no minimum. (0,1) is a local and absolute maximum.

Note that $f'(x) = -\frac{1}{2} \cdot x^{-1/2} = \dfrac{-1}{2\sqrt{x}}$

If $\dfrac{-1}{2\sqrt{x}} = 0$ then $-1 = 0 \cdot (2\sqrt{x})$

$-1 = 0$

which is impossible, so $f'(x)$ is never 0. So there are no critical points other than the endpoint at $x = 0$.