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Math 574
Exam 3
Show All Work!

1. Find a recurrence relation along with appropriate initial conditions for the number c_n of n-digit numbers using the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ such that the sum of the digits is even. Simplify the recurrence relation so that no subscript appears more than once.

   Solution: We can begin with 2, 4, 6, or 8 and continue with any of the c_{n-1} sequences of length $n-1$. Or we can begin with any of 1, 3, 5, 7, or 9 and continue with any of the $9^{n-1} - c_{n-1}$ sequences that have an odd digit sum. Thus for $n \geq 1, c_n = 4c_{n-1} + 5\left(9^{n-1} - x_{n-1}\right)$.

   So we have, $c_0 = 1$, and for $n \geq 1$, $c_n = 5 \cdot 9^{n-1} - c_{n-2}$.

2. (a). Using Inclusion-Exclusion, determine the number of permutations of $1, 2, 3, 4, 5\ 6, 7, 8, 9$ in which each of 1, 2, 3, and 4 is not in its proper position. *Clearly define the sets that you are using*.

   Solution: Define the sets A_i, $i = 1, 2, 3, 4$ to be the number of permutations in which i is in position i.

   Then by inclusion, exclusion:
   $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right| = 9! - 4 \cdot 8! + 6 \cdot 7! - 4 \cdot 6! + 5!$.

   (b). How many functions from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ use each even integer as a value at least once?

   Solution: $9^5 - 4 \cdot 8^5 + 6 \cdot 7^5 - 4 \cdot 6^5 + 5^5$

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3. Let c_n denote the number of strings of length n using the letters a, b, c, d, e that have at least one occurrence of the pattern *ab or* at least one occurrence of *aa*. Find a recurrence relation for c_n along with appropriate initial conditions. Justify your work! Explain how you got your relation.

Solution: We can begin with *b, c, d, e, ac, ad, ae, aa,* or *ab*.

If we start with *b, c, d*, or *e* there are c_{n-1} ways to continue.
If we start with *ac, ad*, or *ae*, there are c_{n-2} ways to continue.

If we start with *aa* or *ab*, then we can continue with *any string* of length $n-2$.

Counting the possibilities in each case gives,

$c_1 = 0$, $c_2 = 2$, and for $n \geq 3$, $c_n = 4c_{n-1} + 3c_{n-2} + 2 \cdot 5^{n-2}$.

4. (a). How many permutations of the digits 1, 2, 3, 4, 5, 6, 7 have exactly 3 numbers in their proper position? [Simplify your answer to an integer.]

Solution: $\binom{7}{3} D_4 = 35 \times 9 = 315$.

(b). How many derangements are there of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 10 such that every even integer is in an even position? [Simplify your answer to an integer.]

Solution: We can arrange the even numbers in D_5 ways, and then after that, arrange the odd numbers also in D_5 ways. So we get, $D_5^{\ 2} = 44^2 = 1936$..

(c). Simplify $p_n - p_{n-1}$ to a simplified fraction (in lowest terms) in terms of n.
[p_n is the probability that a random permutation is a derangement.] Justify your answer.

Solution: $p_n - p_{n-1} = \dfrac{D_n}{n!} - \dfrac{D_{n-1}}{(n-1)!} = \dfrac{D_n - nD_{n-1}}{n!} = \dfrac{(-1)^n}{n!}$.

Or, $p_n - p_{n-1} = \sum_{k=0}^{n} \dfrac{(-1)^k}{k!} - \sum_{k=0}^{n-1} \dfrac{(-1)^k}{k!} = \dfrac{(-1)^n}{n!}$.

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5. Suppose we are making a walkway using Red, Blue, Yellow and Green blocks. The Red and Blue blocks are 1-inch long and the Yellow and Green blocks are 2-inches long. We are not allowed to use two consecutive Red blocks. Let w_n denote the number of possible walkways of length n that we can form.

   (a). Find a recurrence relation for w_n along with appropriate initial conditions.

   Solution: We can begin with a blue block, a yellow block, or a green block. Or we may begin with a red block followed by a blue block, a red block followed by a green block, or a red block followed by a yellow block. Counting the results gives us,

   $w_0 = 1$, $w_1 = 2$, $w_2 = 5$, and for $n \ge 3$, $w_n = w_{n-1} + 3w_{n-2} + 2w_{n-3}$.

   (b). How many possible walkways are there of length 6?

   Solution:
   $w_3 = 13$
   $w_4 = 32$
   $w_5 = 81$
   $w_6 = 203$

   So there are 203 such walkways.

6. Find a formula for the solution to $a_0 = 1$, $a_1 = 3$, and for $n \ge 2$, $a_n = 6a_{n-1} - 8a_{n-2}$.
   Solution: first find the generating function for the a_k's and then expand by partial fractions to get the final answer of $a_n = \dfrac{2^n + 4^n}{2}$.

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7. How many strings of length 8 using the letters *a, b, c, d* do not contain exactly one occurrence of any letter? i.e., each letter either does not occur at all or it occurs at least twice.
Example: *a b b c c c b b* is bad because there is just one *a*.
and *b c c c c d a c* is bad because *b, a* and *d* all occur just once.
However, *a b b d a a d d* is OK – note that *c* does not occur at all in this string.

Solution: Use Inclusion-Exclusion with the sets defined as
$A_1 -$ denotes all strings with exactly one a.
$A_2 -$ denotes all strings with exactly one b.
$A_3 -$ denotes all strings with exactly one c.
$A_4 -$ denotes all strings with exactly one d.

We get, $4^8 - 4\cdot 8\cdot 3^7 + 6\cdot 8\cdot 7\cdot 2^6 - 4\cdot 8\cdot 7\cdot 6$.

8. Show that if each of the line segments joining 66 points is colored red, blue, green, or yellow, then there must exist some triangle (using the given points as its vertices) all of whose sides have the same color.

Solution: First recall that we proved the following statement in class.

Lemma. If the line segments joining some 17 points are colored red, blue or green then there must exist a triangle all of whose sides have the same color.

Now suppose that the line segments have been colored with the given four colors.
Consider any one of the points and the 65 line segments that join the other points to it. Since these line segments are each of one of four colors, then by the Pigeon Hole Principle, there are some 17 of these line segments that have the same color, say yellow. So now consider the 17 points that are the end vertices of these 17 line segments. If any two of them are joined by a yellow edge, then we would get a yellow triangle. Otherwise by the lemma above, we would get a red, blue or green triangle.

9. How many subsets of $S = \{a, b, c, d, e, f, g, h\}$ intersect each of the sets $\{a, b\}$, $\{c, d\}$, $\{e, f\}$, $\{g, h\}$? i.e., How many subsets of *S* contain at least one of *a* or *b*, at least one of *c* or *d*, at least one of *e* or *f*, and at least one of *g* or *h*?
Example: $\{a, c, d, f, h\}$ is OK but $\{a, b, e, f, g, h\}$ is not since it does not intersect $\{c, d\}$.

Solution: Using Inclusion-Exclusion we get,
$2^8 - 4\cdot 2^6 + 6\cdot 2^4 - 4\cdot 2^2 + 1 = 81$.