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Math 547 – Practice Exam #3

1. (a). Explicitly describe the elements of the field $\mathbb{Q}(\pi)$.
   Solution: The elements of $\mathbb{Q}(\pi)$ are quotients of polynomials in π.

   (b). Explicitly describe the elements of the field $\mathbb{Q}\left(\sqrt[3]{2}\right)$.
   Solution: $\mathbb{Q}\left(\sqrt[3]{2}\right)=\left\{a+b\sqrt[3]{2}+c\sqrt[3]{4} : a,b,c\in\mathbb{Q}\right\}$.

   (c). Give a basis for the field $\mathbb{Q}\left(\sqrt{3},\sqrt{2},i\right)$ over $\mathbb{Q}$.
   Solution: $\left\{1,\sqrt{2},\sqrt{3},\sqrt{6},i,i\sqrt{2},i\sqrt{3},i\sqrt{6}\right\}$.

   (d). Define algebraic extension.
   Solution: An extension field *E* of a field *F* is an algebraic extension if every element of *E* is algebraic over *F*.

2. **Prove**: If $F\subseteq K\subseteq E$ are fields and *K* is a finite extension of *F* and *E* is a finite extension of *K*, then $[E:F]=[E:K][K:F]$.
   Solution: See your notes.

3. Suppose that γ is a zero of $p(x)=x^2+2x+3\in Z_5[x]$ in some extension field *E*.
   Note: $p(x)=x^2+2x+3$ is irreducible in $Z_5[x]$; you need not verify this.
   (a). How many elements are there in $Z_5(\gamma)$? Explain.
   Solution: $Z_5(\gamma)=\left\{a+b\gamma : a,b\in Z_5\right\}$. Moreover, each element of $Z_5(\gamma)$ is uniquely expressible in the form $a+b\gamma$. There are 5 ways to choose the values for each of *a* and *b*. Thus there are 25 elements in $Z_5(\gamma)$.

   (b). Express the product $(1+2\gamma)(3+\gamma)$ in the form $a+b\gamma$, $a,b\in Z_5$.
   Solution: $(1+2\gamma)(3+\gamma)=2+3\gamma$.

   (c). Find an expression (in terms of γ) for the other zero of $p(x)=x^2+2x+3$ in *E*.
   Solution: dividing x^2+2x+3 by $x-\gamma$ gives a quotient of $x+(\gamma+2)$. Thus the other zero is $-(\gamma+2)=4\gamma+3$.

4. Let *D* be an integral domain with $F\subseteq D\subseteq E$ where *F* and *E* are fields and *E* is a finite extension of *F*. Show that *D* is a field.
   Solution: See your notes – this is a problem from your text.

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5. Show directly that $\alpha = \sqrt{i+\sqrt{3}}$ is an algebraic number and determine its degree. Fully justify your answer.
Hint: You may take as given that $\sqrt{i+\sqrt{3}} \notin \mathbb{Q}\left(i,\sqrt{3}\right)$.

Solution: Letting $x = \sqrt{i+\sqrt{3}}$, we get $x^2 = i+\sqrt{3}$.
So, $x^4 = -1+2i\sqrt{3}+3 = 2+2i\sqrt{3} \Rightarrow x^4-2 = 2i\sqrt{3}$. Thus $x^8-4x^4+4=-12$.
This implies that $\alpha = \sqrt{i+\sqrt{3}}$ satisfies the polynomial $p(x) = x^8-4x^4+16$.

It might take some effort to show directly that this polynomial is irreducible and hence that the degree of α is 8.

However, since $\mathbb{Q}\left(\sqrt{3}\right) \subseteq \mathbb{R}$, $i \notin \mathbb{Q}\left(\sqrt{3}\right)$. Thus i satisfies the irreducible polynomial $x^2+1 \in \mathbb{Q}\left(\sqrt{3}\right)[x]$ and so $\left[\mathbb{Q}\left(i,\sqrt{3}\right):\mathbb{Q}\left(\sqrt{3}\right)\right]=2$.
Now it follows easily that $\left[\mathbb{Q}\left(i,\sqrt{3}\right):\mathbb{Q}\right] = \left[\mathbb{Q}\left(i,\sqrt{3}\right):\mathbb{Q}\left(\sqrt{3}\right)\right]\left[\mathbb{Q}\left(\sqrt{3}\right):\mathbb{Q}\right]=4$.
Also, $\left[\mathbb{Q}\left(\sqrt{i+\sqrt{3}}\right):\mathbb{Q}\left(i,\sqrt{3}\right)\right]=2$, since $\alpha = \sqrt{i+\sqrt{3}} \notin \mathbb{Q}\left(i,\sqrt{3}\right)$ and α satisfies $p(x) = x^2-\left(i+\sqrt{3}\right)$.

Thus, $\left[\mathbb{Q}\left(\sqrt{i+\sqrt{3}}\right):\mathbb{Q}\right] = \left[\mathbb{Q}\left(\sqrt{i+\sqrt{3}}\right):\mathbb{Q}\left(i,\sqrt{3}\right)\right]\left[\mathbb{Q}\left(i,\sqrt{3}\right):\mathbb{Q}\right]=8$.

6. Given that π is transcendental, show that $\sqrt{\pi}$ cannot be algebraic of degree at most 2.

Solution: Suppose that $\sqrt{\pi}$ is algebraic of degree at most 2. Then there is a polynomial $p(x) = ax^2+bx+c \in \mathbb{Q}[x]$ such that $p\left(\sqrt{\pi}\right)=0$.
So we have, $0 = p\left(\sqrt{\pi}\right) = a\pi + b\sqrt{\pi}+c \Rightarrow a\pi + c = -b\sqrt{\pi}$. Thus squaring both sides we get.
Hence for the polynomial $p(x) = a^2x^2+\left(2ac-b^2\right)x+c^2$, $p(\pi)=0$. So, π would be algebraic of degree at most 2. ⊗

Note: A simple, but more wordy, extension of this idea shows that $\sqrt{\pi}$ is not algebraic of any degree; i.e., $\sqrt{\pi}$ is transcendental.

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7. Suppose that $p(x) \in F[x]$ is irreducible of degree n and that α is a zero of $p(x)$ in some extension field E. Thus $p(x)$ is the minimal polynomial for α.
Let $S = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Show that S is linearly independent in $F(\alpha)$.
Note: Argue directly, you may not use the fact that S is a basis for $F(\alpha)$.

Solution: Suppose that there were elements $b_0, b_1, \ldots, b_{n-1} \in F$ not all zero such that $\sum_{i=0}^{n-1} b_i \alpha^i = 0$. Then $q(x) = \sum_{i=0}^{n-1} b_i x^i$ would be a non-zero polynomial of degree less than that of $p(x)$ and $q(\alpha) = 0$. This contradicts $p(x)$ being the minimal polynomial for α.

8. Let α be algebraic in E over F and suppose that $p(x)$ is its minimal polynomial. Then show that if $f(x) \in F[x]$ with $f(\alpha) = 0$, then $p(x)$ divides $f(x)$.

Solution: See your notes.