Math 546
Final Exam Review Problems

Generally speaking, the best things to study for the final exam are the old exams, practice exams, and exam review sheets.

However, to make you life just a bit easier, these review sheets contains the material that is *most* relevant for the final exam.

Generic Problems: There will be several short problems of some of these types.

- Show that a given set H is a subgroup of a group G.
- Show that a given subgroup H is a normal subgroup of G.
- Show that a given function $\phi : G \to H$ is a homomorphism and determine its kernel and be able to determine if it is 1-1 and/or onto.
- Find the inverse of a given element of a group G, the order of such an element or the identity of G and other basic computations within a group.
- Express a permutation as a product of disjoint cycles.
- Express a permutation as a product of transpositions.
- Perform elementary computations involving permutations.
- Perform elementary computations involving the elements of $U(n)$, Z_n and $GL(2,R)$
- Determine the number of elements of S_n that have a particular order.
- Determine the cyclic subgroups of a group.
- Determine the left cosets of a subgroup of a group.

Definitely not on the Exam
There will not be any remainder problems.
There will not be any problems involving infinite cardinals or rational numbers.

Specific Problems
Be able to work problems similar to those below. At least four problems similar to these will appear on the final exam. Most of these have appeared already on a problem set, exam, or practice exam.

1. Let $f: G \to H$ be a homomorphism of the group G onto the group H. Given that G is Abelian, show that H is abelian. *Indicate precisely* where you used the fact that G is Abelian and where you used that f is a homomorphism.

2. Suppose that A and B are subgroups of the group G and that B is a normal subgroup of G.
(a). Show that $AB = BA$. (b). Show that $H = AB$ is a normal subgroup of G.

3. Let $f: G \to H$ be a homomorphism. Show that f is 1-1 if and only if $\ker(f) = \{e_G\}$.

4. Prove: If G/Z is cyclic, then G is Abelian.

5. Cayley's Theorem guarantees that any group G is isomorphic to a subgroup of S_G. Moreover, the *proof* of Cayley's Theorem provides an explicit isomorphism from G to S_G. For the group $G = Z_{12}$, what element of S_G corresponds to the element 3 of G under this isomorphism? Express your answer as a product of disjoint cycles.

6. (a). If A and B are subgroups of the group G, then $A \cap B$ is a subgroup of G.

   (b). If A and B are subgroups of the group G, and the orders of A and B are relatively prime, then $A \cap B = \{e\}$.

   (c). If the elements a and b of the group G have relatively prime order and $a^k = b^k$ for some integer k, then $a^k = b^k = e$.

7. Suppose that G is an Abelian group and that a and b are distinct elements of G having order 2. Show that $|G|$ is a multiple of 4.
   Hint: What are the values of *a(ab)*, *b(ab)*, *(ab)(ab)*?

8. (a). If the element a in the group G has order k, and $a^m = e$, then k divides m.
   Hint: Let $m = kq + r,\ 0 \le r < d$.
   (b). If G is a group of order n and a is an element of G, then $a^n = e$.

9. If G is a group and $x^2 = e$ for every x in G, then G is abelian.

10. Suppose that G contains exactly one element a of order 2. Show that a belongs to the center of G. Hint: for any x in G consider $\left(xax^{-1}\right)^2$.

11. (a). Let H is a normal subgroup of the group G. Show that if G/H has an element of order k, then so does G.

    (b). Suppose that H is a homomorphic image of the group G. Show that if H has an element of order k, then so does G.

12. If G is a cyclic group, then G is Abelian.

13. Let H be a subgroup of the group G and define $x \equiv y \bmod H \Leftrightarrow x^{-1}y \in H$. Verify that this relation is an equivalence relation. Explain clearly where the properties of H being a subgroup are used.

14. Prove: If G is a group of prime order, then G is cyclic.

15. (a). Prove: Let $\phi : G \to H$ be a homomorphism and $K = \ker\phi$. Then $K \triangleleft G$.
(b). Prove: If $K \triangleleft G$, then $K = \ker\phi$ for some homomorphism $\phi : G \to H$.

16. Suppose that $A \triangleleft G$ and $B \triangleleft G$ and $A \cap B = \{e\}$, where *e* is the identity of *G*. Show that for all $a \in A,\ b \in B$, we have $ab = ba$.
Hint: $ab = ba \Leftrightarrow aba^{-1}b^{-1} = e$.

17. How many generators of the cyclic group Z_n are there?
Is this the same as the number of generators of any other cyclic group of order *n*?

18. Explain why the table below cannot be completed to that of a semigroup.

*	a	b	c	d	e	f
a	b					
b		c				
c			d			
d				e		
e					f	
f						a

19. The table below is that of a partially filled semigroup.
What is the value of $a * b$?

*	*a*	*b*	*c*	*d*
a	*a*		*d*	
b				
c				*b*
d				*c*

20. The table below is the partially filled table for a commutative group on 5 elements. Fill in the missing portions of the table.

*	a	b	c	d	f
a	a				
b		d		f	
c					
d					
f					

21. What is the order of *U*(5000)?

22. Note that $\phi(5) = \phi(8) = \phi(10) = \phi(12) = 4$. Are the groups $U(5),\ U(8),\ U(10),\ U(12)$ isomorphic to one another?

23. The binary structure $\left(R^{+} - \{1\}, \circ\right)$ where $x \circ y = x^{\ln y}$ for all *x, y* in *R*, is a group (you do not need to verify this). What is the inverse of the number e^2?

24. (a). Suppose that $\phi : S_3 \to G$ is a homomorphism. Explain why it is that if we know the value of ϕ on each of $(1,\ 2),\ (1,\ 3),$ and $(2,\ 3)$, then we know all the values of ϕ.

(b). Suppose that $\phi : Z \to H$ is a homomorphism and $\phi(1) = h \in H$. Explain how knowing just this one value of ϕ completely determines $\phi : Z \to H$.

25. (a). Suppose that H is a subgroup of G and o(H) = 12. Suppose that K is a subgroup of G and o(K) = 18. Also $|H \cap K| \geq 4$. What is the value of $|H \cap K|$ (and why)?

(b). Suppose that G is a group of order 20. Could the Centralizer of G contain exactly three elements? Explain.

(c). Suppose that G is a finite group and H is a subgroup of G and K is a subgroup of H. Show that $[G:K] = [G:H] \cdot [H:K]$.

26. Suppose that n and m are relatively prime integers and x and y are elements of the group G such that $x^n = y^n$ and $x^m = y^m$. Show that $x = y$.
Hint: If n and m are relatively prime then there exist integers r and s such that $rn + sm = 1$.

27. If G is a group such that for all a and b in G, $(ab)^2 = a^2b^2$, then G is Abelian.

28. Let $f : R \to R^+$ be defined by $f(x) = e^x$, give the definition of an operation $*$ on R so that f will be an isomorphism from $(R,*)$ to $(R^+, +)$. What is the value of $0 * 0$?