Name Solutions

Directions: Complete the following problems. Show and explain all of your work for full credit. Partial credit will be given for work showing progress towards a solution. Please write legibly and coherently. The use of calculators are not permitted on this test.

1. Compute the following:

(a) $g'(y)$ if $g(y) = \int_3^y \sqrt{1+t^3}\, dt$

$$g'(y) = \sqrt{1+y^3}$$

(b) $\frac{d}{dx}\left(\int_0^{\frac{1}{x}} \tan^{-1} t\, dt\right)$

$$= \tan^{-1}\left(\tfrac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x^2}\tan^{-1}\left(\tfrac{1}{x}\right)$$

(c) $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx$

$$\boxed{\begin{array}{ll} u = \ln x & \to \ln(e^4) = 4 \\ du = \frac{1}{x}dx & \ln(e) = 1 \end{array}}$$

$$= \int_1^4 \frac{1}{\sqrt{u}} du = \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1} \Big|_{u=1}^{u=4}$$

$$= 2\left(\sqrt{4} - \sqrt{1}\right)$$

$$= 2.$$

(d) $\int_0^\pi \theta \sin(3\theta)\, d\theta$ $= -\frac{\theta}{3}\cos(3\theta)\Big|_{\theta=0}^{\theta=\pi} + \frac{1}{3}\int_0^\pi \cos(3\theta)\,d\theta$

$$\boxed{\begin{array}{ll} u = \theta & dv = \sin(3\theta)d\theta \\ du = d\theta & v = -\frac{1}{3}\cos(3\theta) \end{array}}$$

$= -\frac{\pi}{3}\cdot(-1) + \frac{1}{9}\sin(3\theta)\Big|_{\theta=0}^{\theta=\pi}$

$= \frac{\pi}{3} + 0 - 0 = \frac{\pi}{3}.$

(e) $\int_0^{\pi/2} \sin^3(x)\cos^2(x)\, dx$ $= \int_0^{\pi/2} (1-\cos^2\theta)\cos^2\theta \sin\theta\, d\theta$

$$\boxed{\begin{array}{ll} u = \cos\theta & \rightsquigarrow \cos(0) = 1 \quad \cos(\frac{\pi}{2}) = 0 \\ du = -\sin\theta\, d\theta & \end{array}}$$

$= -\int_1^0 (1-u^2)u^2\,du = \int_0^1 u^2 - u^4\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5\Big|_{u=0}^{u=1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$

(f) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

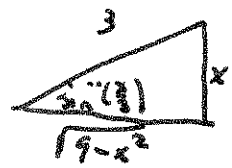
$$\boxed{\begin{array}{ll} \frac{x}{3} = \sin\theta & x^2 = 9\sin^2\theta \\ dx = 3\cos\theta\, d\theta & \theta = \sin^{-1}\left(\frac{x}{3}\right) \end{array}}$$

$$= \int \frac{3\sqrt{1-\left(\frac{x}{3}\right)^2}}{x^2}\,dx$$

$$= 3\int \frac{\cos\theta}{9\sin^2\theta}\cdot 3\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\, d\theta = \int \csc^2\theta - 1\, d\theta = -\cot\theta - \theta$$

$$= -\cot\left(\sin^{-1}\left(\frac{x}{3}\right)\right) - \sin^{-1}\left(\frac{x}{3}\right)$$



$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

(g) $\displaystyle\int \frac{2x+1}{(x+2)(x-1)}\,dx$

$$= \int \frac{A}{x+2} + \frac{B}{x-1}\,dx$$ where $A(x-1) + B(x+2) = 2x+1$ for all x

$x = -2$: $-3A = -3$, so $A = 1$

$x = 1$: $3B = 3$, so $B = 1$

Thus the integral is $\displaystyle\int \frac{1}{x+2}\,dx + \int \frac{1}{x-1}\,dx$

$$= \ln|x+2| + \ln|x-1| + C$$

$$= \ln|(x+2)(x-1)| + C.$$

(h) $\int \frac{x^3 - x^2 - x + 2}{x^3 - 3x^2 + 2x}\, dx$

$$\begin{array}{r} 1 \\ x^3 - 3x^2 + 2x \overline{\big)\, x^3 - x^2 - x + 2} \\ -(x^3 - 3x^2 + 2x) \\ \hline 2x^2 - 3x + 2 \end{array}$$

$$= \int 1 + \frac{2x^2 - 3x + 2}{x^3 - 3x^2 + 2x}\, dx = x + \int \frac{2x^2 - 3x + 2}{x(x-2)(x-1)}\, dx$$

$$= x + \int \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1}\, dx$$, where

$$A(x-2)(x-1) + Bx(x-1) + Cx(x-2) = 2x^2 - 3x + 2$$

$x = 0$: $2A = 2 \to A = 1$

$x = 2$: $2B = 4 \to B = 2$

$x = 1$: $-C = 1 \to C = -1$. Thus, the integral is

$$x + \int \frac{1}{x} + \frac{2}{x-2} - \frac{1}{x-1}\, dx = x + \ln|x| + 2\ln|x-2| - \ln|x-1| + C$$

2. Determine if the following integral converges or diverges. If it converges, give its value.

$$\int_2^\infty \frac{1}{x \ln x}\, dx$$

$$= \lim_{N \to \infty} \int_2^N \frac{1}{x \ln x}\, dx = \lim_{N \to \infty} \int_{\ln 2}^{\ln N} \frac{1}{u}\, du$$

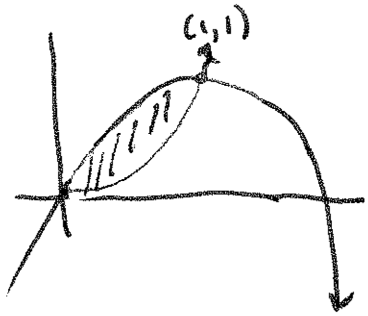
$u = \ln x \to \ln 2, \ln N$

$du = \frac{1}{x}\, dx$

$$= \lim_{N \to \infty} \ln|u| \Big|_{u = \ln 2}^{u = \ln N} = \lim_{N \to \infty} \ln(\ln N) - \ln(\ln 2)$$

$= \infty$. Thus the integral diverges

3. Find the area of the region enclosed by the curves $y = x^2$ and $y = -(x-1)^2 + 1$



The area is

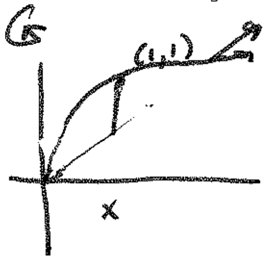
$$= \int_0^1 -(x-1)^2 + 1 - x^2 \, dx$$

$$= \int_0^1 -2x^2 + 2x \, dx = 2\int_0^1 x - x^2 \, dx$$

$$= 2\left(\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=1} \right) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2 \cdot \frac{1}{6}$$

$$= \frac{1}{3}.$$

4. Find the volume of the symmetric solid obtained by rotating the area enclosed by $y = x$ and $y = \sqrt{x}$ about the y-axis.



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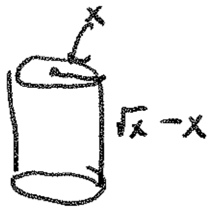

Area is $2\pi x(\sqrt{x}-x)$

So volume is $\displaystyle\int_0^1 2\pi x(\sqrt{x}-x)\,dx = 2\pi\int_0^1 x^{3/2} - x^2\,dx$

$\displaystyle = 2\pi\left(\frac{1}{5/2}x^{5/2} - \frac{1}{3}x^3\Big|_{x=0}^{x=1}\right) = 2\pi\left(\frac{2}{5}-\frac{1}{3}\right) = \frac{2\pi}{15}.$

Alternate Solution:

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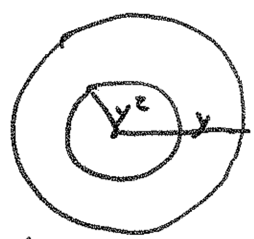

Area is $\pi y^2 - \pi y^4$

So volume is $\displaystyle\int_0^1 \pi y^2 - \pi y^4\,dy$

$\displaystyle = \pi\left(\frac{1}{3}y^3 - \frac{1}{5}y^5\right) = \pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2\pi}{15}.$

5. Calculate the average value of the function $f(x) = \dfrac{2x}{(1+x^2)^2}$ on the interval $[0, 2]$.

The average is $\dfrac{1}{2-0}\displaystyle\int_0^2 \frac{2x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_1^5 \frac{1}{u^2}\,du$

$u = 1+x^2$
$du = 2x\,dx$

$$= \frac{1}{2}\left(\frac{1}{-2+1}u^{-2+1}\Big|_{u=1}^{u=5}\right) = \frac{1}{2}\left(-\frac{1}{5}-(-1)\right)$$

$$= \frac{1}{2}\cdot\frac{4}{5} = \frac{2}{5}.$$

6. Calculate $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

$$= \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^{\infty} xe^{-x^2}dx = \lim_{N\to\infty}\left(\int_{-N}^{0} xe^{-x^2}dx + \int_0^{N} xe^{-x^2}dx\right)$$

$$\boxed{u = -x^2 \quad du = -2x\,dx \quad -\tfrac{1}{2}du = x\,dx}$$

$$= \lim_{N\to\infty}\left(-\int_{-N^2}^{0}\frac{e^u}{2}du - \int_0^{-N^2}\frac{e^u}{2}du\right)$$

$$= \lim_{N\to\infty} -\frac{1}{2}\left(e^u\Big|_{u=-N^2}^{u=0} + e^u\Big|_{u=0}^{u=-N^2}\right)$$

$$= -\frac{1}{2}\lim_{N\to\infty}\left(1 - e^{-N^2} + e^{-N^2} - 1\right) = 0 \quad \left(\frac{1}{e^{N^2}} \to 0 \text{ as } n\to\infty\right)$$

Or, notice xe^{-x^2} is odd (why?), and explain why this means the answer is 0.

Bonus: Calculate

$$\int \frac{x-2}{x^2+2x+5}\,dx$$

$$= \int \frac{x-2}{x^2+2x+1\ -1+5}\,dx = \int \frac{x-2}{(x+1)^2+4}\,dx$$

$$= \frac{1}{4}\int \frac{x-2}{\left(\frac{x+1}{2}\right)^2+1}\,dx = \frac{1}{4}\int \frac{(x+1\ -3)}{\left(\frac{x+1}{2}\right)^2+1}\,dx$$

$$= \frac{1}{4}\int \frac{x+1}{\left(\frac{x+1}{2}\right)^2+1}\,dx - \frac{3}{4}\underbrace{\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\,dx}_{u=\frac{x+1}{2}\quad dx=2du}$$

$$u = \left(\frac{x+1}{2}\right)^2+1$$

$$2du = x+1\,dx \quad \left(2\,\frac{x+1}{2}\cdot\frac{1}{2}\right)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du - \frac{3}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\ln|u| - \frac{3}{2}\tan^{-1}(u)$$

$$= \frac{1}{2}\ln\left(\left(\frac{x+1}{2}\right)^2+1\right) - \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$$