7.5

12) $$\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{1}{u^2+u+1}\,du$$

$$\boxed{\begin{gathered} u = x^2 \\ \tfrac{1}{2}\,du = x\,dx \end{gathered}} \qquad = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}\,du$$

$$= \frac{4}{3}\int \frac{1}{\left(\frac{u+\frac{1}{2}}{\sqrt{3}/2}\right)+1}\,du$$

$$\boxed{\begin{gathered} z = \frac{u+1/2}{\sqrt{3}/2} \\ \tfrac{\sqrt{3}}{2}\,dz = du \end{gathered}}$$

$$= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{z^2+1}\,du = \frac{2\sqrt{3}}{3}\tan^{-1}(z)$$

$$= \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{u+1/2}{\sqrt{3}/2}\right) = \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{x^2+\frac{1}{2}}{\sqrt{3}/2}\right)+C$$

15) $$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{1-x^2}\right)^3}$$

$x = \sin\theta \to \theta = \sin^{-1}x$

$dx = \cos\theta\, d\theta$

$$= \int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \int \sec^2\theta\, d\theta$$

$$= \tan\theta = \tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}} + C.$$

18) $$\int \frac{e^{2t}}{1+e^{4t}}\,dt$$

try $u = e^{2t}$, use $e^{4t} = \left(e^{2t}\right)^2$.

22) $$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$$

try $u = \ln x$ then ...?

or try $\ln x = \tan\theta$ if you are clever. (why?)

23) $\int_0^1 (1+\sqrt{x})^8 dx$

multiply this all out ...?

or $u = 1+\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}} dx = \frac{1}{2(u-1)} dx$$

so $dx = 2(u-1)du$

Thus $\int_0^1 (1+\sqrt{x})^8 dx = 2\int_1^2 u^8(u-1)\, du$

$$= 2\int_1^2 u^9 - u \, du = 2\left(\frac{1}{10}u^{10} - \frac{1}{2}u^2 \Big|_{u=1}^{u=2}\right)$$

$$= 2\left(\frac{1}{10}2^{10} - 2 - \frac{1}{10} + \frac{1}{2}\right) = \frac{2^{11}}{10} - 4 - \frac{1}{5} + 1$$

$$= \frac{2^{11}}{10} - 3 - \frac{1}{5} = \frac{2^{11} - 30 - 2}{10} = \frac{2048 - 32}{10}$$

$$= \frac{2016}{10} = \frac{1008}{5}.$$

24) $\int \ln(x^2-1)\,dx$

Try to make use of $\ln(AB) = \ln A + \ln B$.

Read back to how we computed $\int \ln(u)\,du$.

32) $\int \frac{\sqrt{2x-1}}{2x+3}\,dx$

$$\boxed{\begin{array}{l} u = \sqrt{2x-1} \rightarrow u^2 = 2x-1 \rightarrow u^2+4 = 2x+3 \\ du = \frac{1}{2\sqrt{2x-1}}\,dx \rightarrow 2u\,du = dx \end{array}}$$

$$= \int \frac{2u^2}{u^2+4}\,du = \frac{2}{4}\int \frac{u^2}{(\frac{u}{2})^2+1}\,du$$

$$\frac{u}{2} = \tan\theta \quad \theta = \tan^{-1}\left(\frac{u}{2}\right)$$

$$du = 2\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{4\tan^2\theta}{\sec^2\theta}\,2\sec^2\theta\,d\theta = 4\int \tan^2\theta\,d\theta$$

Now by parts $w = \tan^2\theta$ $dz = d\theta$

$\int w\,dz = \ldots.$

or: $2\int \frac{u^2}{u^2+4}\,du = 2\int 1 + \underbrace{\frac{4}{u^2+4}}_{\text{same as above.}}\,du = \ldots.$

$$\begin{array}{r} 1 \\ u^2+4\overline{)\,u^2} \\ \underline{u^2+4} \end{array}$$

63) $\int \frac{\sin(2x)}{1+\cos^4 x}\,dx = 2\int \frac{\sin x\cos x}{1+\left(\cos^2 x\right)^2}\,dx$

$$\boxed{\begin{aligned} u &= \cos^2 x \\ du &= -2\cos x\sin x\,dx \end{aligned}}$$

$= -\int \frac{1}{1+u^2}\,du = -\tan^{-1}(\cos^2 x) + C.$

65) $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx = \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}\,dx$

$= \int (x+1)^{1/2} - x^{1/2}\,dx$

$= \frac{1}{3/2}(x+1)^{3/2} - \frac{1}{3/2}x^{3/2} = \frac{2}{3}\left((x+1)^{3/2} - x^{3/2}\right) + C.$