Math 554
Handout #4 – 2/13/96

Defn. A subset $\mathcal{O} \subseteq \mathbb{R}$ is called *open* if for each $x \in \mathcal{O}$ there is an ϵ-neighborhood $N_\epsilon(x)$ contained in $\mathcal{O}$.

Homework #5 (Due Wednesday, Feb.13)

1. Prove that each open interval (c, d) is an open set.

Theorem. The following holds true for open subsets of $\mathbb{R}$:

a.) Both $\mathbb{R}$ and $\emptyset$ are open.

b.) Arbitrary unions of open sets are open.

c.) Finite intersections of open sets are open.

Note. Arbitrary intersectons of open sets need not be open.

a.) If $\mathcal{O}_n := (-1/n, 1/n)$, then $\bigcap\limits_{n=1}^{\infty} \mathcal{O}_n = \{0\}$.

b.) If $\mathcal{O}_n := (-1/n, 1 + 1/n)$, then $\bigcap\limits_{n=1}^{\infty} \mathcal{O}_n = [0, 1]$.

c.) If $\mathcal{O}_n := (-1/n, 1)$, then $\bigcap\limits_{n=1}^{\infty} \mathcal{O}_n = [0, 1)$.

Note. Suppose X is a set and $\mathcal{T}$ is a collection of substs of X with the properties

a.) Both X and $\emptyset$ belong to $\mathcal{T}$,
b.) $\mathcal{T}$ is closed under arbitrary unions,
c.) $\mathcal{T}$ is closed under finite intersections,

then (X,T) is called a *topological space* and $\mathcal{T}$ is called a *topology* for X. Moreover, each $\mathcal{O} \in \mathcal{T}$ is a *neighborhood* for each of their points.

Defn. A set $C \subseteq \mathbb{R}$ is called *closed* if its complement is open in $\mathbb{R}$.

Example. Each of the following is an example of a closed set:

a.) Each closed interval $[c, d]$ is a closed subset of $\mathbb{R}$.

b.) The set $(-\infty, d\,] := \{x \in \mathbb{R} |\ x \leq d\}$ is a closed subset of $\mathbb{R}$.

c.) Each singleton set $\{x_0\}$ is a closed subset of $\mathbb{R}$.

d.) The *Cantor set* is a closed subset of $\mathbb{R}$.

To construct this set, start with the closed interval $[0, 1]$ and recursivley remove the open middle-third of each of the remaining closed intervals ...

At the n-th stage, we have 2^n closed intervals each of length $(\frac{1}{3})^n$:

Stage 0: $[0, 1]$

Stage 1: $[0, \frac{1}{3}]$ $[\frac{2}{3}, 1]$

Stage 2: $[0, \frac{1}{9}]$ $[\frac{2}{9}, \frac{3}{9}]$ $[\frac{6}{9}, \frac{7}{9}]$ $[\frac{8}{9}, 1]$

$\vdots$

This finite union of closed intervals is closed. The Cantor set is the intersection of this (decreasing or nested) sequence of sets and so is also closed. Later, we will hopefully see that it has many other interesting properties.

Homework #6 (Due Monday, Feb.19)
Prove each of the following:

a.) Both $\mathbb{R}$ and $\emptyset$ are closed sets.

b.) Arbitrary intersections of closed sets are closed.

c.) Finite unions of closed sets are closed.

d.) $\left\{0, 1, \dfrac{1}{2}, \dfrac{1}{3}, \dots, \dfrac{1}{n}, \dots\right\}$ is a closed set.
