Math 554
1/29/96

Chapter 3 deals with limits and the topology of $\mathbb{R}$. First we recall the concept of induction.

Theorem. (Principle of Mathematical Induction.) Suppose that a statement $\mathbf{p(n)}$ is defined for each natural number n. If

1. $\mathbf{p(1)}$ is true
2. (($\mathbf{p(n)}$ true) $\Longrightarrow$ ($\mathbf{p(n+1)}$ true)) is a true statement for each $n \in \mathbb{N}$,

then $\mathbf{p(n)}$ is a true statement for each natural number n.
Proof. Suppose to the contrary that the set $B := \{n \in \mathbb{N} | \ \mathbf{p(n)} \text{ false}\}$ is not empty. Notice that $1 \notin B$. Let N be the smallest element of B (possible since you can take a minimum of a finite set of integers), then set $n := N - 1$. Observe by assumption (1) that $n \in \mathbb{N}$, and by the definition of B that $\mathbf{p(n)}$ is true. By assumption (2), it follows that $\mathbf{p(n+1)}$ is true. But $n + 1 = N$. Contradiction. Hence B must be empty. □

Example. These will useful in our study of convergence. Both are proved by induction.

1. $\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}$, if $r \neq 1$.
2. $1 + na \leq (1 + a)^n$, if $a > 0$ & $n \in \mathbb{N}$. (Bernoulli's inequality)

Defn. If $\epsilon > 0$ an *ϵ-neighborhood of a* is defined to be the set

$$N_\epsilon(a) := \{x \in \mathbb{R} | \ |x - a| < \epsilon\}.$$

Notice that $N_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

Defn. A *sequence* of real numbers is defined to be a mapping from the natural numbers $\mathbb{N}$ to the reals and is denoted by $a_1, a_2, a_3, \ldots$ or by $\{a_n\}_{n=1}^{\infty}$. The following definitions are used throughout the course:

1. $\{a_n\}$ is *bounded*, if $|a_n| \leq K$, for all $n \in \mathbb{N}$.
2. $\{a_n\}$ is *convergent to a*, denoted by $\lim_{n\to\infty} a_n = a$, if each ϵ-nbhd of a contains all but a finite number of terms of the sequence. We also use the shorter notation $a_n \to a$ when there is no ambiguity on the indices.

Example. The following are examples of sequences:

1. $1/2, 1/3, 1/4, \cdots$
2. $1, r, r^2, r^3, \cdots$
3. $1,\ 1+r,\ 1+r+r^2,\ 1+r+r^2+r^3, \cdots$

Lemma. $\lim_{n\to\infty} a_n = a$ if and only if

$$\textbf{for every } \epsilon > 0, \textbf{there exists } N \in I\!N \textbf{so that if } n \geq I\!N, \textbf{then } |a_n - a| < \epsilon.$$

In short hand this reads '$\forall \epsilon > 0,\ \exists N = N(\epsilon) \in I\!N \ni n \geq I\!N(\epsilon) \implies |a_n - a| < \epsilon$.'

Proof. Notice that if a statement is true except for at most a finite number of terms, then there is a a largest integer for which it is not true. Take N to be that integer's successor. □

Example.

1. $\lim_{n\to\infty} \frac{1}{n} = 0.$

   Proof. Use the Archimedean Principle.

2. $\lim_{n\to\infty} \frac{3n^2 - 1}{n^2 + n + 25} = 3.$

   (Hint: For a given $\epsilon > 0$, use $N := \max\{76, 4N_1\}$ where N_1 is the 'cutoff' for Example 1, i.e. any integer larger than $1/\epsilon$)

3. If $|r| < 1$, then $r^n \to 0$.

   Proof. If $r = 0$, then the conclusion follows straight away. Suppose that $0 < |r| < 1$, then if $b := 1/|r| - 1$ we see that $b > 0$ and $|r| = 1/(1+b)$. By Bernoulli's inequality, $|r^n|^{-1} = (1+b)^n \geq 1 + nb$. Inverting this inequality gives $|r^n - 0| \leq 1/(1+nb)$. By example 1, pick N so that $1/n < b\epsilon$ if $n \geq N$. Hence,

   $$|r^n - 0| \leq \frac{1}{1+nb} < \frac{1}{nb} < \epsilon, \quad \text{if } n \geq N. \quad \square$$

4. $\lim_{n\to\infty} a_n = 1/(1-r)$, if $a_n := 1 + r + r^2 + \cdots + r^n$ and $|r| < 1$.

   Proof. If $r = 0$, the conclusion follows immediately. We may suppose then that $0 < |r| < 1$. In this case, we use the identity above, i.e.

   $$a_n := \sum_{j=0}^{n} r^n = \frac{1 - r^{n+1}}{1 - r}$$

to see that

$$a_n - a = -r^{n+1}/(1-r)$$

where $a := 1/(1-r)$. Now, given $\epsilon > 0$, by example 3 there is an N_0 such that $n \geq N_0$ implies $|r^n| < (\frac{1-|r|}{|r|})\epsilon$. Combined with the displayed equation, this gives $|a_n - a| < \epsilon$ if $n \geq N_0$. $\square$

Theorem. If $\lim_{n\to\infty} a_n$ exists, then it is unique.
Proof. Suppose that $\lim_{n\to\infty} a_n = A_1$ and $\lim_{n\to\infty} a_n = A_2$ and that $A_1 \neq A_2$. Set $\epsilon := |A_1 - A_2|/2$. Now $\epsilon > 0$ so there exists N_1, such that if $n \geq N_1$ then $|a_n - A_1| < \epsilon$. Since the sequence converges to A_2, we also have that there exists N_2, such that if $n \geq N_2$ then $|a_n - A_2| < \epsilon$. Let $N := N_1 + N_2$, then N is larger than both N_1 and N_2 and so

$$|A_1 - A_2| \leq |A_1 - a_N| + |A_2 - a_N| < 2\epsilon = |A_1 - A_2|,$$

which gives a contradiction. $\square$

Theorem. Each convergent sequence is bounded.
Proof. Suppose that $\lim_{n\to\infty} a_n = a$. Let $\epsilon := 1$, then there is an integer N such that $a_n \in N_\epsilon(a)$ if $n \geq N$. This means that $a - 1 < a_n < a + 1$, if $n \geq N$. If $M := \max\{a+1, a_1, a_2, \ldots, a_{N-1}\}$ and $m := \min\{a-1, a_1, a_2, \ldots, a_{N-1}\}$, then

$$m \leq a_n \leq M, \quad \text{for all } n. \quad \square$$

Note. Not every bounded sequence is convergent. For example, the sequence $a_n := (-1)^n$ is bounded, but the sequence is not convergent. To see this take $\epsilon = 1$.