MATH 554/703I - ANALYSIS I
TEST 3 – NOVEMBER 20, 2001

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|---|---|
| 1 | (20 *pts*) |
| 2 | (10 *pts*) |
| 3 | (15 *pts*) |
| 4 | (15 *pts*) |
| 5 | (10 *pts*) |
| 6 | (15 *pts*) |
| 7 | (15 *pts*) |
| | |

**Name:** ____________________________

**Directions:** To receive credit, you must justify your statements unless otherwise stated. Answers should be provided in complete sentences.

1. Give an example of each of the following and (very) briefly justify your answer:

   (a) A bounded set of real numbers that is not compact.

   $(0,1]$ is bounded, but not closed & $\therefore$ not compact.

   (b) A connected set of real numbers that is not compact.

   $(0,1]$ is an interval & $\therefore$ connected, but again not compact.

   (c) A real-valued continuous function that does not satisfy the Extreme Value Theorem.

   $f(x) = \frac{1}{x}$, $0 < x$   $f$ is continuous on its domain, but not bounded.

   or $f(x) = x$, $0 \leq x < 1$

   (d) A real-valued continuous function that does not satisfy the Intermediate Value Theorem.

   $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$   then $f$ is continuous on its domain but $\nexists\ \xi \in \operatorname{dom} f \ni f(\xi) = \frac{1}{2}$.

2. a. State the Bolzanno-Weierstrass property for a subset of real numbers. $K$.

   Each infinite subset $A$ of $K$ has a limit point which belongs to $K$.

   b. State the sequential compactness property for a subset of real numbers. $K$

   Each sequence in $K$ has a subsequence which converges to a member of $K$.

3. a.) Define open cover for a set.

$\mathcal{C}$ is an open cover for a set $K$ if $\mathcal{C}$ is a collection of open sets and $K \subseteq \bigcup_{O \in \mathcal{C}} O$.

b.) Define what it means for a set to be compact.

A set $K$ is compact if each open cover of $K$ has a finite subcover which covers $K$.

c.) Suppose $K$ is compact and $f : K \to \mathbb{R}$ is continuous. Prove that $f[K]$ is compact.

proof Let $\mathcal{C}$ be an open cover of $f[K]$. Define

$$\tilde{\mathcal{C}} := \{ f^{-1}(O) \mid O \in \mathcal{C} \}.$$

Each $\tilde{O} \in \tilde{\mathcal{C}}$ is open (rel. to $K$) since $\exists\, O$ open $\ni$ $\tilde{O} = f^{-1}[O]$ & $f$ is continuous.

Since $\mathcal{C}$ is a cover for $f[K]$, then $\tilde{\mathcal{C}}$ is a cover for $K$. $K$ is compact, $\tilde{\mathcal{C}}$ is an open cover for $K$, so has a finite subcover $\{\tilde{O}_1, \tilde{O}_2, \ldots, \tilde{O}_n\}$.

But then $O_i \ni \tilde{O}_i = f^{-1}[O_i]$, $(1 \le i \le n)$ forms a finite subcover of $\mathcal{C}$ which covers $f[K]$. $\therefore$ $f[K]$ is compact. ∎

4. State and sketch a proof of the Heine-Borel theorem.

Either use the proof in the course notes or if you prefer to use a proof indicated in the reference text, then you must fill in some details as follows:

Thm If $a<b$, then the interval $[a,b]$ is compact.

The proof given in class is replicated in the notes posted on the web. The proof below is one (with completed details) that a good portion of the class wrote.

pf Let $A = \{a \le x \mid [a,x]$ has a finite subcover from $\mathcal{C}\}$. Since $a \in A$, then $A$ is nonempty. If $A$ is not bounded, then $[a,b]$ has a finite subcover from $\mathcal{C}$ since $A$ is an interval & in this case $A = [a, \infty)$.

If $A$ is bounded, then let $\gamma := \sup A$. If $\gamma > b$, then done since $b \in A$. If $\gamma \le b$, then $\mathcal{C}$ an open cover for $[a,b] \Rightarrow \exists$ $O_0 \in \mathcal{C} \ni \gamma \in O_0$. $O_0$ open $\Rightarrow \exists\, \varepsilon > 0 \ni N_\varepsilon(\gamma) \subseteq O_0$. Since $\gamma$ is the least upper bound of $A$ & $\varepsilon > 0$, $\exists\, x \in A \ni$

$$\gamma - \varepsilon < x \le \gamma.$$

Since $x \in A$ $\exists$ a finite subcover $O_1, \ldots, O_n \in \mathcal{C}$ covering $[a,x]$.

But $\bigcup_{j=0}^{n} O_j \supseteq [a, \gamma + \frac{\varepsilon}{2}] \Rightarrow \gamma + \varepsilon/2 \in A$ ✱ $\gamma = \sup A$. Therefore $\gamma \le b$ is impossible. ∎

5. Prove that each closed and bounded subset of real numbers is a compact set.

Suppose $C$ is closed & bounded. Since $C$ is bounded $\exists M \in \mathbb{R}$ $\ni$ $|x| \leq M$, $\forall x \in C$. Hence $C \subseteq [-M, M] =: K$. By the Heine-Borel thm $K$ is compact. But closed subsets of compact sets are also compact, so $C$ is compact.

6. a. Define "disconnection" for a set $A$.

A <u>disconnection</u> $(A_1, A_2)$ for a set $A$ satisfies

$$A = A_1 \cup A_2 \quad ; A_1, A_2 \neq \phi$$

with $A_1, A_2$ open relative to $A$.

b. Define what it means for a set $A$ to be a connected subset of real numbers.

$A$ is connected means $A$ has <u>no</u> disconnections.

c. Show that a subset $A$ of real numbers is connected implies that A is an interval.

If $A$ is connected, but not an interval, then $\exists\, a_1, a_2 \in A$ and $a \notin A$ with $a_1 < a < a_2$. Define

$$A_1 = (-\infty, a) \cap A$$
$$A_2 = (a, \infty) \cap A,$$

then $a_1 \in A_1$, $a_2 \in A_2$, $A_1 \cap A_2 = \phi$, $A = A_1 \cup A_2$.

Hence $A$ is <u>not</u> connected. ※ ▨

7. State and prove the **Intermediate Value Theorem**

Thm Suppose $f:[a,b]\to\mathbb{R}$ is continuous and that $x_1, x_2 \in [a,b]$. If $\eta$ is between $f(x_1)$ and $f(x_2)$, then $\exists\ \xi$ between $x_1$ and $x_2$ so that $\eta = f(\xi)$.

proof WLOG $x_1 \le x_2$ and set $I = [x_1, x_2]$. Since $I$ is connected & $f$ is continuous, then the image $f[I]$ is connected. But $f[I] \subseteq \mathbb{R}$ so $f[I]$ is an interval $J$. By hypothesis $\eta$ is between $f(x_1)$ & $f(x_2)$, both of which belong to the interval $J$. Hence $\eta \in f[I]$. This just means $\exists\ \xi \in I \ni f(\xi) = \eta$. $\blacksquare$