Math 554 HW #5 Solutions Oct. 18, 2001

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#6 $a_1 := 1$, $a_n := 3 - \frac{1}{a_{n-1}}$ $(n \geq 2)$. Prove that $a_n \uparrow$ and bounded.

proof By induction. One way is to first prove

(q_n) $2 \leq a_n \leq 3$, $(n \geq 2)$.

Obviously true for $n=2$. Assume true for n, then $\frac{1}{3} \leq \frac{1}{a_n} \leq \frac{1}{2}$ and $2\frac{1}{2} \leq 3 - \frac{1}{a_n} \leq 2\frac{2}{3}$, so (q_n) holds for $n+1$.

(p_n) $a_n \leq a_{n+1}$, $n \geq 2$.

True for $n=2$, since $a_2 = 2$, $a_3 = 2\frac{1}{2}$. Assume p_n is true, then $-\frac{1}{a_n} \leq -\frac{1}{a_{n+1}}$ (since $a_n > 0$, all n.)

and so
$$3 - \frac{1}{a_n} \leq 3 - \frac{1}{a_{n+1}}$$
or
$$a_{n+1} \leq a_{n+2}.$$
Therefore p_{n+1} is true. ▨

Note: We know a_n must converge to $a = \sup\{a_n\}$. By properties of limits $3 - \frac{1}{a_n} \to 3 - \frac{1}{a}$ as $n \to \infty$ so
$$a = 3 - \frac{1}{a} \Rightarrow a = \frac{3+\sqrt{5}}{2}.$$

#9(b) Required to use the definition of convergence (i.e. ε, N ---).

Soln $a_n = \frac{2n}{3n+2}$ and $a = \frac{2}{3}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \varepsilon$. If $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N}$ and so
$$|a_n - a| = \left|\frac{2n}{3n+2} - \frac{2}{3}\right| = \left|\frac{-4}{(3n+2)(3)}\right|$$
$$< \frac{4}{9}\frac{1}{n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \quad ▨$$

#11 (a) By the reverse triangle inequality $||x_n| - |L|| \leq |x_n - L|$ ...

(b) Let $x_n = (-1)^n$, then $|x_n| \to 1$ as $n \to \infty$, but $\{x_n\}$ does not converge.

(c) Suppose $\varepsilon > 0$. Since $\lim_{n\to\infty} |x_n| = 0$, then $\exists N \in \mathbb{N} \ni n \geq N \Rightarrow ||x_n| - 0| < \varepsilon$. But $||x_n| - 0| = |x_n| = |x_n - 0|$. So for $n \geq N$, $|x_n - 0| < \varepsilon$. ▨

Extra Credit

HW#5 (Cont.)

#17| Suppose $\{x_k\}$, $\{y_k\}$, & $\{r_k\}$ are sequences in $\mathbb{R}$ with $\lim_{n\to\infty} r_k = 0$. Suppose further $0 < |y_k - x_k| < r_k$, all $k \in \mathbb{N}$.

(a) Let $x_k = k$ & $y_k = k + r_k/2$, then

$$0 < r_k/2 = |y_k - x_k| < r_k$$

but $\{x_k\}$ & $\{y_k\}$ are not bnded, so are not convergent.

(b) If $\lim_{k\to\infty} x_k = L$, then

$$|y_k - L| = |x_k - L + y_k - x_k|$$

$$\leq |x_k - L| + |y_k - x_k|.$$

Now apply the hypothesis $(x_k \to L$ & $r_k \to 0)$. $\blacksquare$

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#2 For $n \in \mathbb{N}$, let $x_n = \sum_{k=n+1}^{2n} \frac{1}{k}$. Prove $\{x_n\}$ converges.

Proof We show $\{x_n\}$ is monotone increasing and bounded from above. Clearly, $0 \leq x_n \leq n \cdot \frac{1}{n+1}$ $(\frac{1}{k} \downarrow)$

≤ 1 for all n.

That is $\{x_n\}$ is bounded. To prove $x_n \uparrow$, we observe that

$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$$

so $x_n < x_{n+1}$. $\blacksquare$

Extra Credit

#3| One very similar to this was done in class.