Assignment #2

1. Suppose $A \subseteq \mathbb{R}$ is nonempty and that L is a lower bound for A, then A has a greatest lower bound λ.

<u>proof</u> Define $B = \{b \mid b = -a, \text{ for some } a \in A\}$, then if we set $M = -L$ we can see that M is an <u>upper bound</u> for B; indeed if $b \in B$, then $\exists\, a \in A \ni b = -a$. But $L \le a$ so $-a \le -L = M$. $B \ne \emptyset$ (since $A \ne \emptyset$) so B has a least upper bound, call it γ. Set $\lambda = -\gamma$, then λ is a lower bound for A since γ is an upper bound for B

$$(b \le \gamma,\ \forall b \in B) \iff (\lambda \le a,\ \forall a \in A).$$

If L is any lower bound for A, then $-L$ is an upper bound for B. γ is lub $\Rightarrow$ $\gamma \le -L$. But then $L \le -\gamma = \lambda$. Therefore λ is the largest of all lower bounds $\blacksquare$

2. Show '$<$' satisfies the trichotomy property.

<u>proof</u> $\mathbb{P}$ satisfies the trichotomy property

exactly one of the following holds

(*) $\quad c > 0$, $c = 0$, or $0 > c$.

Given $a, b \in \mathbb{R}$ & define $c = b - a$. Applying (*) to c gives that

either $b - a > 0$, $b - a = 0$, or $0 > b - a$

That is

either $b > a$ <u>or</u> $b = a$ <u>or</u> $a > b$. $\blacksquare$

3. Transitive Property:

If $a < b$ <u>and</u> $b < c$, then

$$[0 < (b-a) \text{ and } 0 < (c-b)] \Rightarrow [(b-a) \in \mathbb{P} \text{ and } (c-b) \in \mathbb{P}]$$

The sum of two positive numbers is positive by def'n of $\mathbb{P}$

so $c - a = (c-b) + (b-a) \in \mathbb{P}$. Therefore $a < c$. $\blacksquare$

(Continued)

Assignment # 2 (cont.)

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\# 8(a) (*) Prove that $\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|$, $\forall n \in \mathbb{N}$.

Proof Prove by induction on n.

case: n=1 $\left|\sum_{k=1}^{1} a_k\right| = |a_1| = \sum_{k=1}^{1} |a_k|$.

induction Step Suppose (*) is true for n, then we need to show the statement is true for n+1. But by the triangle inequality

$$\left|\sum_{k=1}^{n+1} a_k\right| = \left|\left(\sum_{k=1}^{n} a_k\right) + a_{n+1}\right| \leq \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}|$$

$$\leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k|.$$

The last inequality is the induction hypothesis. ∎

\# 9 Suppose $a \neq 0$ & $|x-a| < \frac{|a|}{2}$, for all $x \in S$, $S \neq \emptyset$, then prove $|x| > \frac{|a|}{2}$ $\forall x \in S$.

Proof Suppose $x \in S$ & $|x-a| < \frac{|a|}{2}$, then similar to our proof of the triangle inequality

$$|a| = |x + (a-x)| \leq |x| + |a-x| = |x| + |x-a| < |x| + \frac{|a|}{2}, \quad \forall x \in S.$$

Adding $-\frac{|a|}{2}$ to the inequality $|a| < |x| + \frac{|a|}{2}$, gives $\frac{|a|}{2} < |x|$, $\forall x \in S$. ∎

\# 11 In symbols $\text{dist}(|a|, |b|) \leq \text{dist}(a,b)$ which for $\mathbb{R}$ is just the reverse triangle inequality. $\big||a| - |b|\big| \leq |a-b|$. ∎

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\# 5 If $A \neq \emptyset$ & bounded, then ∃ an upper bound M for A ($x \leq M, \forall x \in A$) and ∃ a lower bound L for A ($L \leq x, \forall x \in A$). Hence

$$L \leq x \leq M, \ \forall x \in A \iff A \subseteq [L, M].$$ ∎