Math 554

Limits and Continuous Functions

Handout #5 – Fall 2008

We need to recall the following:

Earlier Definitions

- A point x_0 is called a *limit point* of a set A if each nbhd of x_0 contains a member of A different from x_0, i.e. for each $\epsilon > 0$, $(N_\epsilon(x_0)\backslash\{x_0\}) \cap A \neq \emptyset$.
- A point $x_0 \in A$ is called an *isolated point of* A if x_0 belongs to A but is not a limit point.

Earlier Theorems

- A set F is closed if and only if it contains all its limit points.
- x_0 is a limit point of a set A if and only if there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x_0$, but $x_n \neq x_0$, $(\forall n \in I\!N)$.

With these ideas we begin the study continuity of functions, which is very naturally framed in a general metric space.

Defn. Suppose that x_0 is a limit point of the domain of a function $f : A \to B$, then f is said to have *a limit* L *as* x *approaches* x_0 if,

$$\forall \epsilon > 0, \exists \delta > 0 \ \ni \ (x \in dom(f) \ \&\ 0 < d_A(x, x_0) < \delta) \Longrightarrow d_B(f(x), L) < \epsilon.$$

In this case, we use the notation,

$$\lim_{x \to x_0} f(x) = L.$$

Defn. Suppose f: $A \to B$, metric spaces. If $x_0 \in A$, then f is said to be *continuous at* x_0 if for each $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in A$ and $d_A(x, x_0) < \delta$, then $d_B(f(x), f(x_0)) < \epsilon$.

Note. Remember that a point $x_0 \in A$ is either an isolated point of A or it is a limit point of A. Considering these two cases separately, the definition for continuity of f at a point can be seen to be equivalent to the following in each of the respective situations:

1. if x_0 is an isolated point of A, then f is automatically continuous.

or

2. if x_0 is a limit point of the domain A, then the condition $\lim_{x \to x_0} f(x) = f(x_0)$ must hold.

Defn. Consider a set $B^* \subseteq B$. A set $\tilde{O} \subseteq B^*$ is called *open relative to* B^* (or briefly, *relatively open*) if $\tilde{O} = \mathcal{O} \cap B^*$ for some open set $\mathcal{O} \subseteq B$. That is to say, $\tilde{O}$ is just the restriction to B^* of an open set in the whole space.

Theorem. Let $f\colon A \to B$, where A, B are metric spaces, then TFAE

a.) f is continuous at each point of its domain,
b.) for each limit point x_0 of the domain A, $\lim_{x \to x_0} f(x)$ exists & equals $f(x_0)$.
c.) for each sequence $x_n \to x_0$, then $f(x_n) \to f(x_0)$ must hold,
d.) $f^{-1}[\mathcal{O}]$ is open for each open subset $\mathcal{O}$ of B.

We will be concentrating on the metric space of real numbers.

Note. The same proof above for continuity at a point x_0 can be used to show the corresponding result for limits holds. The only difference is that f is not required to be defined at x_0. The statement reads as:

Suppose that $f\colon A \to B$ is a real-valued function of a real variable, i.e. $A, B \subseteq \mathbb{R}$. If x_0 is a limit point of the domain of f, then TFAE :

a.) $\lim_{x \to x_0} f(x) = L$,
b.) For every sequence $\{x_n\}$ in the domain of f, if $x_n \to x_0$, then $f(x_n) \to L$.

Corollary. The finite sum, product, or the quotient of continuous functions is each continuous on their respective domains.

Corollary. All polynomials are continuous. Rational functions are continuous on their domains.

Theorem. The composition of continuous functions is continuous.

Example. Each of the following are examples of continuous functions on their respective domains:

1. $f(x) := |x|$
2. $g(x) := \sqrt{x}$
3. $F(x) := \sqrt{\dfrac{x^2 - 2x + 5}{x^3 - 1}}$

Homework #5, due Tuesday Nov 11

1. Suppose that x_0 is a limit point of the domain of $f+g$ and that both f and g have limits at x_0, then prove that

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

2. Suppose that f is defined by

$$f(x) := \begin{cases} 3x+2, & \text{if } -1 \leq x \\ -2x+1, & \text{if } x < -1. \end{cases}$$

Determine at each point whether or not f is continuous. Justify your answer.

3. Determine the domain of $F(x) := \sqrt{\dfrac{x^2-2x+5}{x^3-1}}$ and carefully show that F is continuous at each point of its domain.