Connectedness
Handout #7

Defn. A *disconnection* of a set A is two nonempty sets A_1, A_2 whose disjoint union is A and each is open relative to A. A set is said to be *connected* if it does not have any disconnections.

Example. The set $\left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right)$ is disconnected.

Theorem. Each interval (open, closed, half-open) I is a connected set.
Proof. Let A_1, A_2 be a disconnection for I. Let $a_j \in A_j \neq \emptyset, \quad j = 1, 2$. We may assume WLOG that $a_1 < a_2$, otherwise relabel A_1 and A_2. Consider $E_1 := \{x \in A_1 | x \leq a_2\}$, then E_1 is nonempty with a_2 as an upper bound. Let $a :=$ lub E_1. But $a_1 \leq a \leq a_2$ implies $a \in I$ since I is an interval. First note that by the lemma to the least upper bound property either $a \in A_1$ or a is a limit point of A_1. In either case, $a \in A_1$ since A_1 is closed relative to I. Since A_1 is also open relative to the interval I, then there is an $\epsilon > 0$ so that $N_\epsilon(a) \in A_1$. But then $a + \epsilon/2 \in A_1$ and is less than a_2, which contradicts that a is the lub of E_1. $\square$

Theorem. If A is a connected set, then A is an interval.
Proof. Otherwise, there would be $a_1 < a < a_2$, with $a_j \in A$ and $a \notin A$. But then $\mathcal{O}_1 := (-\infty, a) \cap A$ and $\mathcal{O}_2 := (a, \infty) \cap A$ form a disconnection of A. $\square$

Note. Each open subset of $I\!R$ is the countable disjoint union of open intervals. This is seen by looking at open *components* (maximal connected sets) and recalling that each open interval contains a rational. Relatively open sets (relative with respect to $A \subseteq I\!R$) are just restrictions of these.

Theorem. The continuous image of a connected set is connected. The continuous image of $[a, b]$ is an interval $[c, d]$ where $c = \min\limits_{a \leq x \leq b} f(x)$ and $d = \max\limits_{a \leq x \leq b} f(x)$.

Proof. Any disconnection of the image $f([a, b])$ could be 'drawn back' to form a disconnection of $[a, b]$: if $\{\mathcal{O}_1, \mathcal{O}_2\}$ forms a disconnection for $f(I)$, then $\left\{f^{-1}(\mathcal{O}_1), f^{-1}(\mathcal{O}_2)\right\}$ forms a disconnection for $I = [a, b]$. So it is impossible that $f([a, b])$ is not connected. $\square$

Corollary. (Intermediate Value Theorem) Suppose f is a real-valued function which is continuous on an interval I. If $a_1, a_2 \in I$ and y is a number between $f(a_1)$ and $f(a_2)$, then there exists a between a_1 and a_2 such that $f(a) = y$.

Proof. We may assume WLOG that $I = [a_1, a_2]$. We know that $f(I)$ is a closed interval, say I_1. Any number y between $f(a_1)$ and $f(a_2)$, belongs to $I_1 = f(I)$ and so there is an $a \in [a_1, a_2]$ such that $f(a) = y$. $\square$

Theorem. Suppose that $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point, i.e. there is an $\alpha \in [a, b]$ such that $f(\alpha) = \alpha$.
Proof. Consider the function $g(x) := x - f(x)$, then $g(a) \leq 0 \leq g(b)$. g is continuous on $[a, b]$, so by the Intermediate Value Theorem, there is an $\alpha \in [a, b]$ such that $g(\alpha) = 0$. This implies that $f(\alpha) = \alpha$. $\square$