PROBABILITY
(MATH/STAT 511)
TEST 3 – APRIL 4, 2001

1	(15 *pts*)
2	(15 *pts*)
3	(20 *pts*)
4	(15 *pts*)
5	(15 *pts*)
6	(20 *pts*)

Name: ______________________ **Code (4 letters)** ______

Directions: Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room. Calculators are allowed, but you must show intermediate work for partial credit.

1. Flaws occur on an average of 5 per every 10 meters of computer tape. If these are distributed as a Poisson process, then

   a.) give the probability of exactly 15 flaws in 20 meters of tape. (You may leave your answers in unsimplified form.) This is a Poisson process with $\tilde{\lambda} = \frac{5}{10} 20 = 10$ flaws on average per 20 meters. The probability is given by the Poisson distribution

   $$P(X=15) = e^{-\tilde{\lambda}} \frac{(\tilde{\lambda})^{15}}{15!} = \boxed{e^{-10} \frac{(10)^{15}}{15!}}$$

   b.) what is the probability that no flaws appear in the first 20 meters of tape? (You may leave your answers in unsimplified form.)
   In this case, $\tilde{\lambda}$ = mean rate of # of flaws / 20 meters, so

   $$\tilde{\lambda} = \frac{5}{10} \cdot 20 = 10$$

   $$P(X=0) = e^{-10} \frac{(10)^0}{0!} = \boxed{e^{-10}}$$

2. The probability that a machine produces a defective item is 0.10. Each item is checked as it is produced. Assume that these are independent trials and compute the probability that at least 10 items must be checked to find one that is defective.

   X = # of items checked until first defective is found.
   X has a geometric distribution:

   $$P(X=x) = q^{x-1} p$$

   $$P(X \geq 10) = \sum_{x=10}^{\infty} q^{x-1} p = p q^9 \sum_{j=0}^{\infty} q^j = \frac{q^9 p}{1-q} = q^9$$

   $$= \boxed{(.9)^9}$$

3. Compute the following for the continuous random variable X with probability mass function

$$f(x) := \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) cumulative distribution,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0, & x \le 0 \\ \int_0^x 2(1-y)\,dy, & 0 < x < 1 \\ \int_0^1 2(1-y)\,dy, & x \ge 1 \end{cases}$$

$$= \boxed{\begin{cases} 0, & x \le 0 \\ 1-(1-x)^2, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}}$$

(b) mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\, 2(1-x)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \boxed{\frac{1}{3}}$$

(c) variance.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\, 2(1-x)\,dx =$$

$$= 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$$

$$\sigma^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \boxed{1/18}$$

4. Show that the moment generating function $M(t)$ for the **standard normal** distribution is $e^{\frac{1}{2}t^2}$.

p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$

Soln: $M(t) = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} dx$

$= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx = e^{t^2/2}$. ∎

5. Telephone calls arrive at a support services center according to a Poisson process on the average of two every five minutes. Let X denote the waiting time until the first call that arrives after 9 a.m.

a.) What is the p.d.f. of X and the corresponding required parameters?

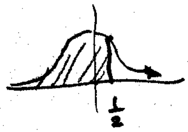
The model for X is an exponential distribution with parameter $\theta = \frac{1}{\lambda}$ min. where $\lambda = 2/5$ per minute.

b.) Compute the probability that X is greater than 6 minutes.

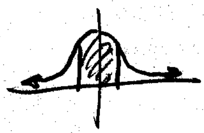
$P(X > 6) = \int_6^{\infty} \frac{2}{5} e^{-2/5 x} dx = \boxed{e^{-12/5}}$

6. Suppose Z is distributed as a standard normal random variable, then compute

a) $P(Z \le \frac{1}{2}) = \Phi\left(\frac{1}{2}\right) = \boxed{.6915}$



b) $P(|Z| \le \frac{1}{2}) = P\left(-\frac{1}{2} \le Z \le \frac{1}{2}\right) = 2\Phi\left(\frac{1}{2}\right) - 1 = \boxed{.3829}$



c) If $Y = u(Z)$ is a random variable defined by $u(z) = 1 + 2z$, then show that $E[Y] = 1$.

$$E[Y] = E[2Z+1] = 2E[Z] + E[1].$$
$$= 2 \cdot 0 + 1 = \boxed{1}$$

d) For this same Y, compute $E[Y^2]$.

$$E[Y^2] = E[(1+2Z)^2] = E[1+4Z+4Z^2]$$
$$= E[1] + 4E[Z] + 4E[Z^2]$$
$$= 1 + 4 \cdot 0 + 4 \cdot 1 = \boxed{5}$$