PROBABILITY
(MATH/STAT 511)
TEST 2 – MARCH 9, 2001

1	(16 *pts*)
2	(10 *pts*)
3	(17 *pts*)
4	(12 *pts*)
5	(15 *pts*)
6	(15 *pts*)
7	(15 *pts*)

Name: ____________________________

Directions: Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room. Calculators are allowed, but you must show intermediate work for partial credit.

1. Define each of the following terms:

a.) *hypergeometric* probability distribution. Describe the typical random variable X which has this as its probability mass function.

$$f(x) = \binom{N_1}{x}\binom{N_2}{n-x} \Big/ \binom{N_1+N_2}{n}, \quad 1 \le n \le N_1+N_2, \quad 0 \le x \le \min(N_1, n), \quad n-x \le N_2$$

X = # of successes in a sample of size n drawn at random without replacement from a population of size $N = N_1 + N_2$, where N_1 = # of successes.

b.) the *expectation* of a random variable, as given in class.

$$E[X] := \sum_{s \in S} X(s) P(\{s\})$$

c.) *variance* of a random variable X.

$$\sigma^2 := E[(X-\mu)^2] \quad \text{where} \quad \mu := E[X].$$

d.) the *moment generating function* for a random variable X.

$$M(t) := E[e^{tX}] = \sum_{x \in \mathcal{S}} e^{tx} f(x)$$

where $f(x) = \sum_{\substack{s \in S \\ X(s) = x}} P(\{s\})$ is the probability mass function.

2. Prove that $\sigma^2 = E[X^2] - E[X]^2$.

$$\sigma^2 = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

linearity

$$= E[X^2] - 2\mu E[X] + \mu^2 E[1]$$

$\mu = E[X]$
$1 = E[1]$

$$= E[X^2] - 2\mu\mu + \mu^2 = E[X^2] - (E[X])^2. \quad \blacksquare$$

3. Suppose the probability that a basketball player makes a free throw is 70% and that each attempt is independent of the rest.

a.) What is the probability that on their fourth attempt, the player makes a free throw?

By independence, on any trial probability $= \boxed{.7}$

b.) If a player has eight attempts, what is the probability that they do not make them all?

This is the complement of the event that they do make them all, so

$1-(.7)^8 \doteq \boxed{.9424}$

c.) If the random variable X is defined as the number of free throws made in the eight attempts, what is the **type** of the probability distribution function of X?

X is Binomial with $n=8$ & $p=.7$

d.) What are the mean and variance of X?

For Binomial

$\mu = np = 5.6$

$\sigma^2 = npq = 1.68$

4. If X is a Binomial distribution with mean $\mu = 12$ and $\sigma^2 = 3$, then compute $P(1 \le X \le 4)$.

First note (as in the homework) that for a Binomial distribution

$$\left.\begin{array}{l} 12 = \mu = np \\ 3 = \sigma^2 = npq \end{array}\right\} \Rightarrow q = \tfrac{1}{4} \Rightarrow \boxed{p = \tfrac{3}{4}}$$

backsubstituting gives $\boxed{n = 16}$

$$P(1 \le X \le 4) = \sum_{x=1}^{4} \binom{16}{x} (.75)^x (.25)^{16-x}$$

$$= B(16, \tfrac{1}{4}; 15) - B(16, \tfrac{1}{4}; 11)$$ ← if you choose to use the tables

# of successes between 1 & 4 ≡ # of failures between 12 & 15

$\doteq \boxed{0.000}$

5. Let X be a discrete random variable with probability mass function

$$f(x) = \frac{4 - |5 - x|}{16}, \quad x = 2, 3, 4, 5, 6, 7, 8.$$

a.) Compute the mean and variance of X.

x	x^2	$f(x)$
2	4	1/16
3	9	2/16
4	16	3/16
5	25	4/16
6	36	3/16
7	49	2/16
8	64	1/16

$E[X] = \frac{80}{16} = \boxed{5}$

$\sigma^2 = E[X^2] - \mu^2 = \frac{440 - 400}{16} = \boxed{\frac{5}{2}}$

b.) Compute the moment generating function for X.

$\sum_{2}^{8} e^{xt} f(x) = \boxed{(e^{2t} + 2e^{3t} + 3e^{4t} + 4e^{5t} + 3e^{6t} + 2e^{7t} + e^{8t})/16}$

6. If the moment generating function of a random variable X is given by

$$M(t) = \frac{1}{12}\left(1 + 2e^{2t}\right)\left(1 + 3e^{-t}\right) = \frac{1}{12}\left(1 + 2e^{2t} + 3e^{-t} + 6e^{t}\right)$$

then compute the

a.) probability mass distribution of X,

x	$f(x)$
-1	3/12
0	1/12
1	6/12
2	2/12

by our theorem

b.) expectation of X.

either

$E[X] = M'(0)$

or directly

$E[X] = (-1)(3/12) + 0(\frac{1}{12}) + 1(\frac{6}{12}) + 2(\frac{2}{12}) = \boxed{\frac{7}{12}}$

7. If the moment generating function of a random variable X is given by

$$M(t) = (.8 + .2e^t)^4,$$

then compute the variance of X.

$\sigma^2 = E[X^2] - (E[X])^2$

where by our theorem,

$E[X] = M'(0) = .8$

$E[X^2] = M''(0) = 1.28$

So

$\sigma^2 = 1.28 - (.8)^2 = \boxed{.64}$