DIFFERENTIAL EQUATIONS
(MATH 242.01)
TEST 1 – SEPTEMBER 20, 2002

Name: ____________________

Directions:
Answer all questions in the space provided. You can also use the back of the facing opposite page if you need more room. You must show intermediate work for partial credit. Calculators are allowed.

1	(10 *pts*)
2	(15 *pts*)
3	(15 *pts*)
4	(15 *pts*)
5	(10 *pts*)
6	(10 *pts*)
7	(15 *pts*)
8	(10 *pts*)

1. Verify that the function $y(x) = x^2 e^x$ satisfies the initial value problem

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^x$$
$$y(0) = 0, \quad \frac{dy}{dx}(0) = 0$$

$y' = (x^2+2x)e^x, \quad y'' = (x^2+4x+2)e^x$

$\Rightarrow y'' - 2y' + y = ((x^2+4x+2) - 2(x^2+2x) + x^2)e^x = 2e^x$ ✓ ODE

$y(0) = 0^2 e^0 = 0$

$y'(0) = (0^2+2\cdot 0)e^0 = 0$ (I.C.) ✓

2. Your stock market investment has a half life of 1 day. Your current investment (financed from your student loan) is \$2,000. You would like to take your best "friend" out to a nice dinner and be able to say it was financed by your market savy. How many days can you procrastinate before cashing in your investment and still have \$70 left for the dinner?

$A(t)$ = money in the account after t days.

$A' = (-\lambda)A, \quad \lambda > 0 \qquad A_0 = A(0) = \$2{,}000$ (I.C.)

$\frac{1}{2} = e^{-\lambda \cdot 1\,\text{day}} \Rightarrow \lambda = \frac{\ln 2}{1\,\text{day}}$ or $A(t) = \$2000\, e^{-\ln 2 \left(\frac{t}{\text{day}}\right)}$

Find t (in 'day' units) so that

$\$70 = \$2000\, e^{-\ln 2 \cdot t}$

$\Rightarrow t = \frac{\ln(200/7)}{\ln 2} = \boxed{4.837 \text{ days}}$

In each of the following four problems, first identify the type of equation and then solve, either explicitly or implicitly, as appropriate.

3. a. Solve for the general solution of

$$y' = \frac{3y}{x} + x^3 e^{2x}$$

$y' + \left(-\frac{3}{x}\right) y = x^3 e^{2x}$ type $\boxed{\text{linear } 1^{st} \text{ order}}$ $P = -\frac{3}{x}$ $Q = x^3 e^{2x}$

Integrating factor $\phi(x) = \exp\left(\int P\,dx\right) = \exp(-3\ln x) = x^{-3}$

$y(x) = \dfrac{\int \phi Q\,dx + c}{\phi} = \dfrac{\int e^{2x}dx + c}{x^{-3}} = \boxed{x^3\left(c + \frac{1}{2}e^{2x}\right)}$ general solution

b. Find a particular solution if y(1)=0.

I.C. $\Rightarrow$ $0 = y(1) = 1^3\left(c + \frac{1}{2}e^{2\cdot 1}\right) \Rightarrow c = -\frac{1}{2}e^2$

or $\boxed{y(x) = \frac{x^3}{2}\left(e^{2x} - e^2\right)}$ particular solution

4. a. Solve for the general solution of

$$y' - 2xy^2 = 0$$

$y' = (2x)(y^2)$ $\boxed{\text{separation of variables}}$ type

$\int \frac{1}{y^2}dy = \int 2x\,dx \Rightarrow -\frac{1}{y} = x^2 + c$

or $\boxed{y(x) = -\frac{1}{x^2+c}}$ general soln

b. Find a particular solution if $y(2) = \frac{1}{5}$.

I.C. $\Rightarrow$ $\frac{1}{5} = y(2) = -\frac{1}{2^2 + c}$ or $c = -5 - 4 = -9$

$\Rightarrow$ $\boxed{y(x) = \frac{1}{9 - x^2}}$ particular solution

5. $(4y^3 - x)y' - y = 3x^2 + cos(\pi x)$

$$(-y-3x^2-\cos\pi x)dx + (4y^3-x)dy = 0$$

M N

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \Rightarrow \boxed{\text{exact equation}}$$

integrate M w.r.t. x to get

$$F(x,y) = -xy - x^3 - \frac{1}{\pi}\sin(\pi x) + g(y)$$

$$(4y^3-x) = \frac{\partial F}{\partial y} = -x + g'(y) \Rightarrow g(y) = y^4 + c$$

$$\therefore \quad F(x,y) = -xy - x^3 - \frac{1}{\pi}\sin(\pi x) + y^4 + c$$

$$\boxed{-xy - x^3 - \frac{1}{\pi}\sin(\pi x) + y^4 = c}$$ implicit solution for y

6. $xyy' + y^2 = x$

$$y' + \left(\frac{1}{x}\right)y = \frac{1}{y}$$ $\boxed{\text{Bernoulli equ with } n=-1.}$ type

let $u = y^{1-n} = y^2$, then $u' = 2yy'$. Substitute back in to get

$$\frac{1}{2y}u' + \left(\frac{1}{x}\right)y = \frac{1}{y} \Rightarrow u' + \left(\frac{2}{x}\right)y^2 = 2 \xRightarrow{\text{use } u=y^2} u' + \left(\frac{2}{x}\right)u = 2$$

transform over to u

linear 1st order in u

$$\phi(x) = x^2$$

$$u = \frac{\int x^2 \cdot 2\,dx + c}{x^2}$$

$$= \frac{\frac{2}{3}x^3 + c}{x^2} = \frac{2}{3}x + \frac{c}{x^2}$$

$u = y^2$

transform back

$$\boxed{y^2 = \frac{2}{3}x + \frac{c}{x^2}}$$

7. Consider the equation $y' = y^2 - 2y$.

a.) Find the critical points and graph the phase line of the equation.

$y' = f(x,y) = y(y-2)$ autonomous

$y' = 0$ when $\left.\begin{array}{l} y=0 \\ y=2 \end{array}\right\}$ critical lines

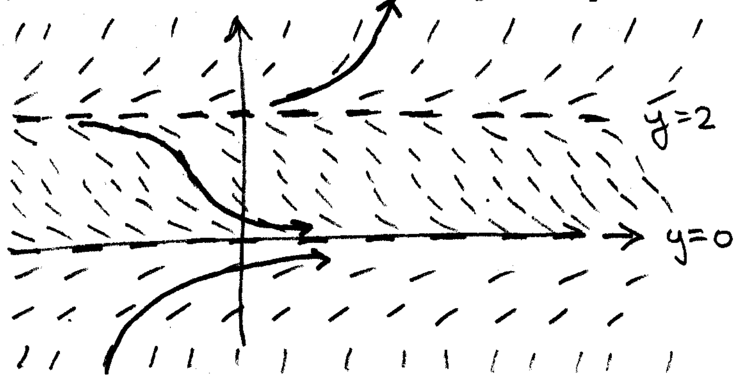
y' ⊕ ⊖ ⊕ 0 2 y

0 2 y

b.) Classify each critical point as an attractor, ~~a repeller, or as semi~~-stable.

$y=2$ repeller
$y=0$ attractor

c.) Sketch the slope field and a couple of representative solutions.



8. On the graph of the flow field, plot some representative solutions.