MATH 142 - CALCULUS II (SECTIONS 11-12)
TEST 3 – APRIL 7, 2005

1	(20 *pts*)
2	(21 *pts*)
3	(20 *pts*)
4	(21 *pts*)
5	(18 *pts*)
EC	(10 *pts*)

Name: ______________________________

Directions: Calculators will be allowed this Test. To receive proper credit however, you must show your intermediate work and *box* your final answer.

1. Compute the first 6 terms of the Taylor series for each of the following functions:

(a) $\frac{1}{2+x}$, about $x_0 = 0$.

n	$f^{(n)}(x)$	$\frac{f^{(n)}(0)}{n!}$
0	$(x+2)^{-1}$	$\frac{1}{2}$
1	$-1(x+2)^{-2}$	$-\frac{1}{4}$
2	$+2(x+2)^{-3}$	$\frac{1}{8}$
3	$-3!(x+2)^{-4}$	$-1/16$
4	$+4!(x+2)^{-5}$	$1/32$
5	$-5!(x+2)^{-6}$	$-1/64$

$$\boxed{\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{32}x^4 - \frac{1}{64}x^5}$$

(b) $\sin(x)$, about $x_0 = \frac{\pi}{2}$.

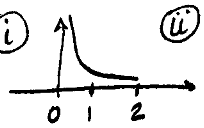
n	$f^{(n)}(x)$	$\frac{f^{(n)}(\pi/2)}{n!}$
0	$\sin x$	$1/0! = 1$
1	$\cos x$	$0/1! = 0$
2	$-\sin x$	$-1/2! = -1/2$
3	$-\cos x$	$-0/3! = 0$
4	$\sin x$	$\frac{1}{4!} = \frac{1}{4!}$
5	$\cos x$	0

$$\sum_{n=0}^{5} \frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n$$

$$\boxed{1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}}$$

2. Compute the following integrals, by (i) sketching the graph, (ii) stating whether they are proper or improper (why), and (iii) whether or not they exist. If they do exist, determine their value.

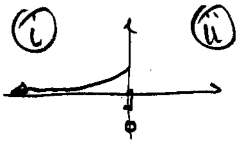
(a) $\int_0^2 x^{-2}\,dx$

(i) 

(ii) Vertical asymptote at $x=0$ $\Rightarrow$ $\boxed{\text{improper}}$

(iii) $\lim_{b\to 0^+} \int_b^2 x^{-2}dx = \lim_{b\to 0^+} -\frac{1}{x}\Big|_{x=b}^{x=2} = \lim_{b\to 0^+}\left(\frac{1}{b}-\frac{1}{2}\right) = +\infty$ $\boxed{\text{DNE as a real number.}}$

(b) $\int_{-\infty}^{0} exp(x)\,dx$

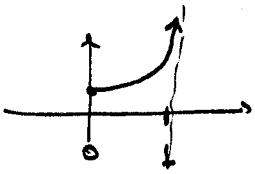
(i) 

(ii) Horizontal asymptote along negative x-axis. Infinite interval of integration $\Rightarrow$ $\boxed{\text{improper}}$

(iii) $\lim_{b\to -\infty} \int_b^0 e^x dx = \lim_{b\to -\infty}\left(e^x\Big|_{x=b}^{x=0}\right) = \lim_{b\to -\infty}(1-e^b) = \boxed{1}$

$\boxed{\text{Exists and equals 1}}$

(c) $\int_0^1 \frac{1}{(1-x)^{\frac{3}{2}}}\,dx$

(i) 

(ii) vertical asymptote at $x=1$ $\Rightarrow$ $\boxed{\text{improper}}$

(iii) $\lim_{b\to 1^-} \int_0^b \frac{1}{(1-x)^{3/2}}dx = \lim_{b\to 1^-}\left(+2(1-x)^{-1/2}\Big|_{x=0}^{x=b}\right) = 2\cdot\lim_{b\to 1^-}\left((1-b)^{-1/2}-1\right)$

$= +\infty$

$\boxed{\text{D.N.E. as a real number}}$

3. Compute the following numerical approximations to $\int_1^5 \frac{x}{x+2}\,dx$

$\Delta x = \frac{5-1}{n} = \frac{1}{2}$

x_k	$y_k = f(x_k)$	Trap.	Simp
1	2/6	1	1
3/2	3/7	2	4
2	4/8	2	2
5/2	5/9	2	4
3	6/10	2	2
7/2	7/11	2	4
4	8/12	2	2
9/2	9/13	2	4
5	10/14	1	1

(a) trapezoidal rule with $n = 8$ total intervals

$$\text{Trap}_8 = \frac{1}{2}\frac{1}{2}\left[\frac{1}{3}+2\left(\frac{3}{7}+\frac{1}{2}+\frac{5}{9}+\frac{3}{5}+\frac{7}{11}+\frac{2}{3}+\frac{9}{13}\right)+\frac{5}{7}\right]$$

$$\cong 2.48020868$$

(b) Simpson's rule with $2n = 8$ total intervals

$$\text{Simpson}_{2n=8} = \frac{1}{3}\cdot\frac{4}{8}\cdot\left[\frac{1}{3}+4\cdot\frac{3}{7}+2\cdot\frac{1}{2}+4\cdot\frac{5}{9}+2\cdot\frac{3}{5}+4\cdot\frac{7}{11}+2\cdot\frac{2}{3}+4\cdot\frac{9}{13}+\frac{5}{7}\right]$$

$$\cong 2.42440522$$

4. For each of the following sequences determine whether it is (i) either eventually increasing or decreasing, (ii) bounded, (iii) convergent. Justify your answers.

(a) $\left\{n + \frac{7}{n}\right\}_{n=1}^{\infty}$

(i) Set $f(x) = x + 7/x$, then $f(n) = a_n$. $f'(x) = 1 - 7/x^2 > 0$ if $x > \sqrt{7}$. So $a_n \nearrow$, if $n \geq 3$. That is a_n $\boxed{\text{increases eventually}}$

(ii) $a_n = n + 7/n > n$, so as $n \to \infty$ a_n is $\boxed{\text{unbounded}}$

(iii) unbounded so not convergent.

For (i) Could also use other tests: $a_{n+1} - a_n = 1 - \frac{7}{n(n+1)} > 0$, if $n \geq 3$.

or $\frac{a_{n+1}}{a_n} = \frac{n^3 + 2n^2 + 8n}{n^2 + 8n + 7} > 1$ if $n \geq 3$.

(b) $\left\{\frac{2n+1}{3n-2}\right\}_{n=1}^{\infty}$

(i) $f(x) = \frac{2x+1}{3x-2} \Rightarrow f'(x) = \frac{-7}{(3x-2)^2} < 0 \Rightarrow$ $\boxed{\{a_n\} \text{ decreases}}$ or $\frac{a_{n+1}}{a_n} = \frac{6n^2+5n-6}{6n^2+5n+1} < 1$, $a_{n+1} - a_n = \frac{-7}{(3n+1)(3n-2)} < 0$

(ii) $0 \leq a_n \leq a_1 = 3$, all n $\boxed{\text{bounded above \& below}}$

(iii) Converges since bounded from below & $\searrow$. Also $\lim\limits_{n\to\infty} \frac{2n+1}{3n-2} = \lim\limits_{n\to\infty}\left(\frac{2 + \frac{1}{n}}{3 - 2/n}\right) = 2/3$.

(c) $\left\{\frac{2^n}{(n+1)!}\right\}_{n=1}^{\infty}$

(i) $\frac{a_{n+1}}{a_n} = \frac{\left(2^{n+1}/(n+1+1)!\right)}{\left(2^n/(n+1)!\right)} = \frac{2}{n+2} < 1$ if $n \geq 1$. $\boxed{a_n \text{ decreases}}$

(ii) $0 \leq a_n \leq a_1 = 1$. $\boxed{a_n \searrow \text{ \& bounded from below}} \Rightarrow$ (iii) $\boxed{a_n \text{ converges}}$

In fact, $a_n = \frac{2}{n+1} a_{n-1} \Rightarrow L \xleftarrow[n\to\infty]{\text{as}} a_n = \left(\frac{2}{n+1}\right) a_{n-1} \to 0 \cdot L$ as $n \to \infty$

So $\boxed{L = 0}$

5. **[EC-10 pts]** Determine the first 15 terms of the Taylor series of the function $x^3 \exp(x^2)$.

Base pt. is $x_0 = 0$.

$$e^u = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^6}{6!} + \text{error}, \quad \text{error} = C \cdot \frac{u^7}{7!}$$

Let $u = x^2$, then

$$x^3 e^{x^2} = x^3\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{12}}{6!}\right) + \text{error}$$

$$= \boxed{x^3 + x^5 + \frac{x^7}{2!} + \frac{x^9}{3!} + \frac{x^{11}}{4!} + \frac{x^{13}}{5!} + \frac{x^{15}}{6!} + O(x^{17})}$$

6. Determine whether or not each of the following series converges. If it converges, then determine its limit.

(a) $\sum_{k=1}^{\infty} \frac{2^{k-2}}{e^k} = \sum_{k=1}^{\infty} (2^{-2})\left(\frac{2}{e}\right)^k = \frac{1}{4} \sum_{k=1}^{\infty} r^k$ where $r = 2/e < 1$

$= \frac{1}{4} \frac{r}{1-r} = \boxed{\frac{1}{2(e-2)}}$

(b) Express 1.8191919... first as a geometric series, then as a rational number.

$x = 1.8 + \frac{1}{10}(.19 + .0019 + .000019 + \cdots)$

$= 1.8 + \frac{19}{10}\left(\frac{1}{100} + \frac{1}{(100)^2} + \frac{1}{(100)^3} + \cdots\right)$, $a = \frac{19}{10}$, $r = \frac{1}{100}$

$= 1.8 + \frac{19}{10}\left(\frac{\left(\frac{1}{100}\right)}{1-\left(\frac{1}{100}\right)}\right) = 1.8 + \frac{19}{990} = \boxed{\frac{1801}{990}}$

(c) $\sum_{k=1}^{\infty} \left(-\frac{2}{3}\right)^{k+1} = \sum_{k=1}^{\infty} (-2/3)^2 (-2/3)^{k-1} = \sum_{m=0}^{\infty} (4/9)(-2/3)^m$

Geometric series with $a = 4/9$ & $r = -2/3$.

Converges since $|r| < 1$.

Converges to $\frac{a}{1-r} = \frac{4/9}{1-(-2/3)} = \boxed{4/15}$