

MAXIMAL SINGULAR INTEGRALS ON L^∞

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The space $BMO(\mathbb{R}^n)$ of functions of bounded mean oscillation consists of those locally integrable functions f on \mathbb{R}^n for which

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

is finite; here the supremum extends over all cubes Q in \mathbb{R}^n (with sides parallel to the coordinate axes) and f_Q denotes the mean value of f over Q : $f_Q = |Q|^{-1} \int_Q f(x) dx$ (cf. [1, p. 141]).

Singular integral operators such as the Riesz transforms (the Hilbert transform when $n = 1$) do not preserve $L^\infty(\mathbb{R}^n)$ but instead carry $L^\infty(\mathbb{R}^n)$ boundedly into $BMO(\mathbb{R}^n)$ (cf. [1, Theorem 1]). The purpose of the present note is to show that the corresponding maximal operators have the same property.

We consider Calderón – Zygmund kernels k , that is, kernels k on \mathbb{R}^n satisfying the following conditions (cf. [2, p. 35]):

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- (i) $\int_{|u| \geq 2|x|} |k(u-x) - k(u)| du \leq M_1 \quad (x \neq 0);$
- (ii) $|k(x)| \leq \frac{M_2}{|x|^n} \quad (x \neq 0);$
- (iii) $\int_{r < |x| < R} k(x) dx = 0 \quad (0 < r < R < \infty),$

for some constants M_1 and M_2 depending only on k .

For each $\epsilon > 0$, let $k_\epsilon(x) = k(x)\chi_{(\epsilon, \infty)}(|x|)$. The singular integral operator K associated with k is defined on $L^\infty(\mathbb{R}^n)$ by the principal value integral

$$(Kf)(x) = \lim_{\epsilon \rightarrow 0} (k_\epsilon * f(x) - k_1 * f(0)),$$

and the corresponding maximal operator \mathcal{K} is given by

$$\mathcal{K}f(x) = \sup_{\epsilon > 0} |k_\epsilon * f(x) - k_1 * f(0)|.$$

Theorem. *If f belongs to $L^\infty(\mathbb{R}^n)$, then $\mathcal{K}f$ belongs to $BMO(\mathbb{R}^n)$ and*

$$\|\mathcal{K}f\|_{BMO(\mathbb{R}^n)} \leq c \|f\|_{L^\infty(\mathbb{R}^n)}$$

where c is a constant depending only on n , M_1 and M_2 .

Proof. It is enough to show that for each cube Q in \mathbb{R}^n there exists α such that

$$(1) \quad \int_Q |\mathcal{K}f - \alpha| dx \leq c \|f\|_{L^\infty(\mathbb{R}^n)} |Q|.$$

Fix Q and let \tilde{Q} be the cube concentric with Q but with diameter $5n$ times as large. Set $b = f\chi_{\tilde{Q}}$ and $g = f - b$, and choose $\alpha = |Q|^{-1} \int_Q \mathcal{K}g dx$. A simple computation using the subadditivity of \mathcal{K} gives

$$(2) \quad \int_Q |\mathcal{K}f - \alpha| dx \leq \int_Q \mathcal{K}b dx + \int_Q |\mathcal{K}g - \alpha| dx.$$

We estimate separately the two terms on the right.

Since k is a Calderón – Zygmund kernel standard arguments show that \mathcal{K} is bounded on $L^2(\mathbb{R}^n)$. Hence, using the Cauchy – Schwarz inequality, we have

$$(3) \quad \begin{aligned} \int_Q \mathcal{K}b \, dx &\leq |Q|^{\frac{1}{2}} \|\mathcal{K}b\|_{L^2(\mathbb{R}^n)} \leq c|Q|^{\frac{1}{2}} \|b\|_{L^2(\tilde{Q})} \leq \\ &\leq c|Q|^{\frac{1}{2}} |\tilde{Q}|^{\frac{1}{2}} \|b\|_{L^\infty(\mathbb{R}^n)} \leq c|Q| \|f\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

To estimate the second term on the right of (2), note that

$$\int_Q |\mathcal{K}g - \alpha| \, dx = 2 \int_Q [\mathcal{K}g - \alpha]_+ \, dx = 2 \int_\Omega (\mathcal{K}g - \alpha) \, dx,$$

where $\Omega = \{x \in Q: \mathcal{K}g(x) > \alpha\}$. Hence, in view of (2) and (3), in order to establish (1) it will suffice to show that

$$(4) \quad \mathcal{K}g - \alpha \leq c \|f\|_{L^\infty(\mathbb{R}^n)} \quad (x \in \Omega).$$

To this end, let $G_\epsilon(z) = |k_\epsilon * g(z) - k_1 * g(0)|$. Fix x in Ω and choose any $\epsilon > 0$ for which $G_\epsilon(x) > \alpha$. Since α is the mean value of $\mathcal{K}g$ over Q , there is some point y in Q for which $G_\epsilon(y) < \alpha$. Hence

$$\begin{aligned} G_\epsilon(x) - \alpha &\leq |G_\epsilon(x) - G_\epsilon(y)| \leq |k_\epsilon * g(x) - k_\epsilon * g(y)| \leq \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{(\tilde{Q})^c} |k_\epsilon(x-u) - k_\epsilon(y-u)| \, du. \end{aligned}$$

But k_ϵ also satisfies conditions (i), (ii), and (iii) above and with constants independent of ϵ (cf. [2, p. 37]). Hence using (i) to estimate the preceding integral we obtain a uniform bound independent of x, y , and ϵ . This completes the proof.

REFERENCES

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