Annals of Mathematics, **113** (1981), 601-611

# Weak-$L^\infty$ and BMO

By COLIN BENNETT[1], RONALD A. DEVORE[2] and ROBERT SHARPLEY[1]

Dedicated to Professor George G. Lorentz on the occasion of his seventieth birthday

## 1. Introduction

The Marcinkiewicz space weak-$L^p$ properly contains $L^p$ when $0 < p < \infty$ but it coincides with $L^\infty$ when $p = \infty$. Consequently, the Marcinkiewicz interpolation theorem does not directly apply to operators that are unbounded on $L^\infty$. The main purpose of this paper is to construct a rearrangement-invariant space $W$ that will play the role of "weak-$L^\infty$", in the sense that it contains $L^\infty$ and possesses the appropriate interpolation properties. The construction, which is motivated by elementary considerations in the Lions-Peetre real interpolation method, is valid for general measure spaces. However, if the underlying measure space is a cube in $\mathbf{R}^n$, then $W$ has an alternative characterization in terms of the space BMO of functions of bounded mean oscillation.

The space $W$ consists of those measurable functions $f$ for which $f^{**} - f^*$ is bounded $\left(\text{where } f^* \text{ is the decreasing rearrangement of } f \text{ and } f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds\right)$. Although no explicit use will be made of the fact, it is perhaps of some interest to note that the space $W$ so-defined arises via the real interpolation method from the pair $(L^\infty, L^1)$ in exactly the same way that the space weak-$L^1$ arises from the reversed pair $(L^1, L^\infty)$. This and other properties of $W$ are developed in Section 2. In particular, a Marcinkiewicz-type interpolation theorem is established for $W$ and it is shown that this result gives a direct proof of the $L^p$-boundedness of the Hilbert transform and related singular integral operators for all values of $p$ with $1 < p < \infty$. With these properties, and the fact that $W$ can be realized as a limit of the familiar spaces weak-$L^p$ as $p \to \infty$, the space $W$ may justifiably be referred to as weak-$L^\infty$.

The relationship between weak-$L^\infty$ and BMO is established in Section 3. A covering argument is used to relate the oscillation of a function $f$ to that

0003-486X/81/0113-3/0601/011 $ 00.55/1

[1] The work of these authors was supported in part by NSF Grant No. MCS-7703666.
[2] The work of this author was supported in part by NSF Grant No. MCS-7722982.

of its decreasing rearrangement $f^*$, and thereby to establish the main result that weak-$L^\infty(Q)$, where $Q$ is a cube in $\mathbf{R}^n$, is precisely the rearrangement-invariant hull of BMO$(Q)$.

In the final section the Hardy-Littlewood maximal operator is shown to be bounded from $W$ into $W$ and from BMO into BMO.

## 2. The space weak-$L^\infty$

The Peetre $K$-functional for the pair $(L^1, L^\infty)$, with respect to an arbitrary $\sigma$-finite measure space $(X, \mu)$, can be explicitly identified as follows:

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s)ds = tf^{**}(t) \qquad (t > 0)$$

(cf. [2, p. 184]). The norm in the Marcinkiewicz space weak-$L^1$ is therefore given in terms of the $K$-functional by

$$(2.1) \qquad ||f||_{\text{weak-}L^1} \equiv \sup_{t>0} tf^*(t) = \sup_{t>0} t\frac{d}{dt}K(f, t; L^1, L^\infty) .$$

If the roles of $L^1$ and $L^\infty$ are now reversed, then a simple computation, together with the identity $K(f, t; L^\infty, L^1) = tK(f, t^{-1}; L^1, L^\infty)$, shows that the functional corresponding to that on the right of (2.1) is simply $\sup_{t>0}[f^{**}(t) - f^*(t)]$.

*Definition* 2.1. Let $W = W(X)$ denote the set of $\mu$-measurable functions $f$ on $X$ for which $f^*(t)$ is finite for all $t > 0$ and for which $f^{**}(t) - f^*(t)$ is a bounded function of $t$. Let

$$(2.2) \qquad ||f||_W = \sup_{t>0}[f^{**}(t) - f^*(t)] \qquad (f \in W) .$$

It is clear that $W$ contains $L^\infty$, and the containment is proper on the interval $(0, 1)$ (or any nonatomic measure space) since $\log(1/t)$, for example, belongs to $W(0, 1)$ but not to $L^\infty(0, 1)$. This logarithmic rate of growth for $f^*$ at the origin is in fact the maximum attainable for any $f$ in $W$. This follows at once from the elementary identity

$$(2.3) \qquad f^{**}(t) - f^{**}(s) = \int_t^s [f^{**}(u) - f^*(u)]\frac{du}{u} \qquad (0 < t \leqq s < \infty)$$

by putting $s = 1$ and using (2.2) to estimate the integrand. But such a growth condition does not characterize $W$, as easy examples show. The fact is that membership in $W$ depends not on the growth of $f^*$ or $f^{**}$ but rather on the growth of the *derivative* of $f^{**}$. In fact, a simple computation gives

$$f^{**}(t) - f^*(t) = -t\frac{d}{dt}(f^{**}(t))$$

at each point of differentiability of $f^{**}$, that is, at each point of continuity of $f^*$. It should also be pointed out that $W$ is not a linear space: there are in fact nonnegative functions in $W$ whose sum is not in $W$. There are also functions $f$ in $W$ such that neither $f_+$ nor $f_-$ belongs to $W$.

When $1 < p < \infty$, it follows from (2.3) (with $s = \infty$) that the functional

$$\left(\int_0^\infty \left(t^{1/p}[f^{**}(t) - f^*(t)]\right)^q \frac{dt}{t}\right)^{1/q} \qquad (0 < q \leqq \infty)$$

is finite if and only if $f$ belongs to the Lorentz space $L^{pq}$. With $q = 1$, this expression converges to $\|f\|_{L^\infty}$ as $p \to \infty$. Thus $L^\infty$ may be regarded in this way as the limit of the Lorentz spaces $L^{p1}$. By the same token the space $W$ is the limit as $p \to \infty$ of the Lorentz spaces $L^{p\infty} = \text{weak-}L^p$. This suggests the following definition.

Recall [10, p. 184] that a sublinear operator $T$ is of *weak type* (1, 1) if it is a bounded map from $L^1$ into weak-$L^1$:

$$\sup_{t>0} t(Tf)^*(t) \leqq c\int_0^\infty f^*(t)dt \qquad (f \in L^1) . \tag{2.4}$$

By analogy, $T$ will be said to be of *weak type* $(\infty, \infty)$ if it is a bounded map from $L^\infty$ into $W$:

$$\sup_{t>0}[(Tf)^{**}(t) - (Tf)^*(t)] \leqq c \sup_{t>0} f^*(t) \qquad (f \in L^\infty) . \tag{2.5}$$

Our interpolation theorem will merely require that (2.4) and (2.5) hold for characteristic functions. Hence, in accordance with the Stein-Weiss terminology [10, p. 197], a sublinear operator $T$ will be of *restricted weak* type (1, 1) (respectively, *restricted weak type* $(\infty, \infty)$) if its domain contains all simple functions and if (2.4) (respectively, (2.5)) holds for all characteristic functions $f = \chi_E$ of sets $E$ of finite measure. The following interpolation theorem is best formulated in terms of the Calderón maximal operator $S$ [3, p. 288]:

$$(Sf)(t) = \frac{1}{t}\int_0^t f(u)du + \int_t^\infty f(u)\frac{du}{u} \qquad (t > 0) .$$

THEOREM 2.2. *Let $T$ be a sublinear operator of restricted weak types* (1, 1) *and* $(\infty, \infty)$. *Then, for all simple functions $f$,*

$$(Tf)^{**}(t) \leqq cS(f^{**})(t) \qquad (t > 0) \tag{2.6}$$

*and*

$$\|Tf\|_{L^p} \leqq c_p\|f\|_{L^p} \qquad (1 < p < \infty) , \tag{2.7}$$

*where $c$ depends only on $T$, and $c_p$ only on $p$ and $T$. In particular, if $T$ is linear, then $T$ has a unique extension to a bounded linear operator on $L^p$ $(1 < p < \infty)$.*

*Proof.* Let $E$ be any $\mu$-measurable subset of $X$ with $0 < s = \mu(E) < \infty$. Let $\chi$ denote the characteristic function of $E$ and let $g = T\chi$. Then the hypotheses on $T$ (cf. (2.4) and (2.5)) give

$$tg^*(t) \leqq cs \qquad (t > 0) \tag{2.8}$$

and

$$g^{**}(t) - g^*(t) \leqq c \qquad (t > 0)\,, \tag{2.9}$$

where $c$ is a constant depending only on $T$. These estimates may be combined to give

$$g^*(t) \leqq 2c\left\{\left(\frac{s}{t} \wedge 1\right) + \log^+\left(\frac{s}{t}\right)\right\} \qquad (t > 0)\,. \tag{2.10}$$

This follows at once from (2.8) if $t \geqq s$. In the remaining case where $0 < t < s$, the estimate (2.9) may be used to estimate the integrand in (2.3) (applied to $g$) to give $g^{**}(t) \leqq g^{**}(s) + c\log(s/t)$, and this yields (2.10) since successive applications of (2.9) and (2.8) show that $g^{**}(s) \leqq g^*(s) + c \leqq 2c$.

The right-hand side of (2.10) is precisely $2cS(\chi^*)(t)$, where $S$ is the Calderón operator. Hence (2.10) may be written in the form

$$(T\chi)^*(t) \leqq 2cS(\chi^*)(t) \qquad (t > 0)\,.$$

An integration of both sides and some further computation now yield the more desirable form

$$(T\chi)^{**}(t) \leqq 2cS(\chi^{**})(t) \qquad (t > 0)\,, \tag{2.11}$$

the point being that the operation $f \to f^{**}$ is subadditive whereas $f \to f^*$ is not. This, together with the sublinearity of $T$, enables us, with standard arguments (cf. [3, pp. 286–287]), to pass from the estimate (2.11) for characteristic functions to the desired estimate (2.6) for all simple functions. The remaining assertions are routine consequences of this one.

The Hilbert transform $H$ may be interpolated directly by the previous theorem. All that is needed is the Stein-Weiss estimate [10, p. 240]

$$(H\chi_E)^*(t) = \frac{1}{\pi}\sinh^{-1}\left(\frac{2|E|}{t}\right) \qquad (t > 0)\,,$$

valid for any subset $E$ of $(-\infty, \infty)$ with finite measure $|E|$. It follows at once from this identity that $H$ is of restricted weak types (1, 1) and $(\infty, \infty)$, and hence that $H$ may be interpolated by Theorem 2.2. The interpolation theorem applies also to the maximal Hilbert transform and, more generally, to the maximal operators associated with arbitrary Calderón-Zygmund singular integrals (cf. [9, p. 35]).

It is worth pointing out that Herz [5] has an interpolation theorem which is somewhat loosely related to ours. The functional $f^{**} - f^*$ is implicit in the proof and it plays a prominent role in some of Herz' applications to martingales. Our interpolation theorem may also be compared with a result of N. M. Riviére [6], to the effect that if $T$ is of weak type (1, 1) and maps $L^\infty$ into BMO, then $T$ is bounded on every $L^p$ with $1 < p < \infty$. In view of Theorem 3.1 of the next section, this result is contained in ours, at least when the underlying measure space is a cube in $\mathbf{R}^n$.

## 3. Weak-$L^\infty$ and BMO

In this section the underlying measure space will be a fixed cube $Q$ (with sides parallel to the coordinate axes) in $\mathbf{R}^n$ with Lebesgue measure. For each integrable function $f$ on $Q$, the *sharp function* of $f$ relative to $Q$ is defined by

$$f_Q^\#(x) = \sup_{Q \supset Q' \ni x} \frac{1}{|Q'|} \int_{Q'} |f(y) - f_{Q'}| dy \qquad (x \in Q) , \tag{3.1}$$

where $f_{Q'} = 1/|Q'| \int_{Q'} f(y)dy$ and the supremum is taken over all cubes $Q'$ that contain $x$ and are contained in $Q$. If $f_Q^\#$ is a bounded function of $x$, then $f$ is said to belong to BMO$(Q)$. The norm is given by

$$||f||_{\mathrm{BMO}(Q)} = \sup_{x \in Q} f_Q^\#(x) . \tag{3.2}$$

It is well-known that BMO can serve as a useful substitute for $L^\infty$ (cf. [4], [6], [7], [8], [11]). The next theorem shows that BMO for a cube $Q$ is intimately connected with $W(Q)$.

THEOREM 3.1. (a) *If $f$ belongs to $L^1(Q)$, then*

$$f^{**}(t) - f^*(t) \leqq c(f_Q^\#)^*(t) \qquad \left(0 < t < \frac{1}{6}|Q|\right), \tag{3.3}$$

*where $c$ is a constant depending only on $n$.*

(b) *The space $W(Q)$ is the rearrangement-invariant hull of* BMO$(Q)$ *in the sense that an integrable function $f$ belongs to $W(Q)$ if and only if $f$ is equimeasurable with some function $g$ in* BMO$(Q)$.

The following covering lemma, which is a variant of Lemma 1.1 in [1], will be needed. The proof is similar so we omit it.

LEMMA 3.2. *Let $\mathcal{O}$ be a relatively open subset of $Q$ such that $|\mathcal{O}| < (1/2)|Q|$. Then there is a family of cubes $Q_j$ ($j = 1, 2, \cdots$) with pairwise disjoint interiors such that*

(i) $|\mathcal{O} \cap Q_j| \leqq 2^{-1}|Q_j| < |\mathcal{O}^c \cap Q_j|$ $\qquad (j = 1, 2, \cdots)$ ;

(ii) $\mathcal{O} \subset \bigcup_{j=1}^\infty Q_j \subset Q$ ;

(iii) $|\mathcal{O}| \leqq \sum_{j=1}^\infty |Q_j| \leqq 2^{n+1}|\mathcal{O}|$.

*Proof of Theorem* 3.1. Since $|f|_Q^\# \leqq 2f_Q^\#$, it is enough to establish (3.3) for nonnegative $f$. In that case, fix $t$ with $0 < t < (1/6)|Q|$ and let

$$E = \{x \in Q: f(x) > f^*(t)\}\ , \qquad F = \{x \in Q: f_Q^\#(x) > (f_Q^\#)^*(t)\}\ .$$

Then $|E \cup F| \leqq 2t$ so there is a relatively open subset $\mathcal{O}$ of $Q$ with $|\mathcal{O}| \leqq 3t$ and $E \cup F \subset \mathcal{O} \subset Q$. In particular $|\mathcal{O}| \leqq (1/2)|Q|$ so by Lemma 3.2 there is a covering $\{Q_j\}_{j=1}^\infty$ of $\mathcal{O}$ satisfying conditions (i), (ii), and (iii) above. Now

$$\begin{aligned} t\{f^{**}(t) - f^*(t)\} &= \int_E \{f(x) - f^*(t)\}dx = \textstyle\sum_{j=1}^\infty \displaystyle\int_{E\cap Q_j} \{f(x) - f^*(t)\}dx \\ &\leqq \textstyle\sum_j \displaystyle\int_{Q_j} |f(x) - f_{Q_j}|dx + \textstyle\sum_j |E \cap Q_j|\{f_{Q_j} - f^*(t)\} \\ &= A + B,\ \text{say}\ . \end{aligned}$$

If $\Sigma'$ denotes the sum over those indices $j$ for which $f_{Q_j} > f^*(t)$, then

$$B \leqq \Sigma' |E \cap Q_j|\{f_{Q_j} - f^*(t)\} \leqq \Sigma' |\mathcal{O} \cap Q_j|\{f_{Q_j} - f^*(t)\}\ .$$

Hence, by (i),

$$B \leqq \Sigma' \int_{\mathcal{O}^c \cap Q_j} \{f_{Q_j} - f^*(t)\}dx \leqq \Sigma' \int_{Q_j} |f_{Q_j} - f(x)|\,dx \leqq A\ ,$$

where the middle inequality holds because $f(u) \leqq f^*(t)$ on $\mathcal{O}^c$. This, together with the preceding estimate, gives

$$t\{f^{**}(t) - f^*(t)\} \leqq 2A\ . \tag{3.4}$$

Now observe from (i) that each $Q_j$ meets $F^c$ in at least one point, say $x_j$. Then $f_Q^\#(x_j) \leqq (f_Q^\#)^*(t)$ because of the way $F$ is defined, and so

$$A = \textstyle\sum_j |Q_j| \displaystyle\left\{\frac{1}{|Q_j|}\int_{Q_j} |f(x) - f_{Q_j}|dx\right\} \leqq \textstyle\sum_j |Q_j| f_Q^\#(x_j) \leqq \sum_j |Q_j|(f_Q^\#)^*(t)\ .$$

Hence, by (iii),

$$A \leqq 2^{n+1}|\mathcal{O}|(f_Q^\#)^*(t) \leqq 2^{n+1}(3t)(f_Q^\#)^*(t)\ ,$$

and this together with (3.4) establishes (3.3).

For part (b), note first that if $t \geqq (1/6)|Q|$, then

$$f^{**}(t) - f^*(t) \leqq f^{**}\Big(\frac{1}{6}|Q|\Big) \leqq 6f^{**}(|Q|) = \frac{6}{|Q|}\int_Q |f(x)|\,dx\ .$$

The inequality (3.3) may be used to estimate $f^{**} - f^*$ in the case $t < (1/6)|Q|$, so together these estimates give

$$\|f\|_{W(Q)} \leqq c\Big(\|f\|_{\mathrm{BMO}(Q)} + \frac{1}{|Q|}\int_Q |f(x)|\,dx\Big)\ . \tag{3.5}$$

This shows that $\mathrm{BMO}(Q)$ is contained in $W(Q)$ and hence, since $W(Q)$ is rearrangement-invariant, that every function $f$ equimeasurable to a

BMO$(Q)$-function $g$ must lie in $W(Q)$.

It will suffice to prove the converse for the unit cube $Q = I^n$ (where $I = [0, 1]$) since a linear change of variables reduces the general case to this one. But then if $f \in W(I^n)$, the function

$$g(x) = f^*(x_1) \qquad (x = (x_1, x_2, \cdots, x_n) \in I^n)$$

is equimeasurable with $f$, and for any subcube $R = \prod_{i=1}^n [r_i, r_i + \alpha]$ of $I^n$,

$$\begin{aligned} \frac{1}{|R|}\int_R |g(x) - f^*(r_1+\alpha)|\,dx_1 \cdots dx_n \\ &= \frac{1}{\alpha}\int_{r_1}^{r_1+\alpha}[f^*(t) - f^*(r_1+\alpha)]dt \\ &\leqq \frac{1}{r_1+\alpha}\int_0^{r_1+\alpha}[f^*(t) - f^*(r_1+\alpha)]dt \\ &= f^{**}(r_1+\alpha) - f^*(r_1+\alpha) \leqq \|f\|_{W(Q)} . \end{aligned}$$

Hence $g$ belongs to BMO$(Q)$ and the proof is complete.

The preceding theorem fails when $Q$ is replaced by all of $\mathbf{R}^n$ since BMO$(\mathbf{R}^n)$ contains functions (such as $\log|x|$) which are unbounded at infinity and hence have decreasing rearrangements which are identically infinite. However, the theorem does contain "local" information pertinent to BMO$(\mathbf{R}^n)$. For example, when $f$ is in BMO$(\mathbf{R}^n)$, the inequality (3.3) may be applied to the function $(f - f_Q)\chi_Q$. An integration of both sides produces the basic inequality (4.23) of [1] from which the John-Nirenberg lemma follows easily.

## 4. Maximal operators

As in the previous section let $Q$ be a fixed cube in $\mathbf{R}^n$. The Hardy-Littlewood maximal function $M_Q f$ of an integrable function $f$ on $Q$ is given by

$$(M_Q f)(x) = \sup \frac{1}{|Q'|}\int_{Q'} |f(y)|\,dy \qquad (x \in Q),$$

where the supremum is taken over all cubes $Q'$ contained in $Q$ and containing $x$. When $Q$ is replaced by all of $\mathbf{R}^n$, the corresponding operator, defined for all locally integrable $f$ on $\mathbf{R}^n$, will be denoted simply by $M$. The next result shows that such maximal operators are bounded on $W$.

THEOREM 4.1. (a) *If $f$ belongs to $W(Q)$, then so does $M_Q f$ and*

$$\|M_Q f\|_{W(Q)} \leqq c\|f\|_{W(Q)} , \tag{4.1}$$

*where $c$ depends only on the dimension $n$.*

(b) *The same result holds if $Q$ is replaced by $\mathbf{R}^n$ and $M_Q$ by $M$.*

*Proof.* (a) We may assume that $f$ is nonnegative. Fix $t < |Q|$ and let

$$b = \max(f - f^*(t), 0), \qquad g = \min(f, f^*(t)),$$

so $f = b + g$. The weak (1, 1) and strong $(\infty, \infty)$ properties of $M_Q$ give

$$(M_Q f)^*(t) \leqq (M_Q b)^*(t-) + (M_Q g)^*(0+) \leqq ct^{-1}\|b\|_{L^1} + \|g\|_{L^\infty}$$
$$\leqq ct^{-1}\int_0^t [f^*(s) - f^*(t)]ds + f^*(t).$$

Hence $(M_Q f)^*(t)$ is finite and

$$0 \leqq (M_Q f)^*(t) - f^*(t) \leqq c\{f^{**}(t) - f^*(t)\} \qquad (t > 0). \tag{4.2}$$

Now write

$$(M_Q f)^{**} - (M_Q f)^* = [(M_Q f)^{**} - f^{**}] + [f^{**} - f^*] + [f^* - (M_Q f)^*]$$

and

$$(M_Q f)^{**}(t) - f^{**}(t) = \frac{1}{t}\int_0^t [(M_Q f)^*(s) - f^*(s)]ds.$$

Then an application of (4.2) yields

$$(M_Q f)^{**}(t) - (M_Q f)^*(t) \leqq c \sup_{0<s\leqq t}\{f^{**}(s) - f^*(s)\},$$

from which (4.1) follows. Exactly the same proof establishes part (b).

Next we show that $M_Q$ is a bounded operator on BMO$(Q)$. Essentially the same result holds for $\mathbf{R}^n$ except that functions $f$ for which $Mf$ is identically infinite must be ruled out ($f(x) = \log|x|$ is an example).

THEOREM 4.2. (a) *If $f$ belongs to* BMO$(Q)$, *then so does $M_Q f$ and*

$$\|M_Q f\|_{\mathrm{BMO}(Q)} \leqq c\|f\|_{\mathrm{BMO}(Q)}, \tag{4.3}$$

*where $c$ depends only on the dimension $n$.*

(b) *If $f$ belongs to* BMO$(\mathbf{R}^n)$, *and if $Mf$ is not identically infinite, then $Mf$ belongs to* BMO$(\mathbf{R}^n)$ *and*

$$\|Mf\|_{\mathrm{BMO}(\mathbf{R}^n)} \leqq c\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} \tag{4.4}$$

*where $c$ depends only on $n$.*

*Proof.* (a) We may assume that $f$ is nonnegative. Writing $F$ for the maximal function $M_Q f$ of $f$, we thus need to show

$$\frac{1}{|R|}\int_R |F(x) - F_R|dx \leqq c\|f\|_{\mathrm{BMO}(Q)} \tag{4.5}$$

for arbitrary subcubes $R$ of $Q$.

Fix $R$ and let $3R$ denote the cube that is concentric with $R$ and has three times the diameter. Let $\tilde{R}$ be the smallest subcube of $Q$ containing

$(3R) \cap Q$, and for each $x$ in $R$ let

$$F_1(x) = \sup\{f_{\bar R}: \bar R \subset \tilde R \text{ and } x \in \bar R\}\,,$$
$$F_2(x) = \sup\{f_{\bar R}: \bar R \subset Q,\ x \in \bar R, \text{ and } \bar R \cap (Q\backslash \tilde R) \neq \varnothing\}\,.$$

Clearly $F = \max\{F_1, F_2\}$ on $R$ so if

$$\Omega = \{x \in R: F(x) > F_R\},\ \Omega_1 = \{x \in \Omega: F_1(x) \geqq F_2(x)\} \text{ and } \Omega_2 = \Omega\backslash\Omega_1\,,$$

then

$$\frac{1}{|R|}\int_R |F(x) - F_R|\,dx = \frac{2}{|R|}\int_\Omega [F(x) - F_R]dx = \frac{2}{|R|}\sum_{i=1}^2 \int_{\Omega_i}[F_i(x) - F_R]dx\,.$$

Hence (4.5) will be established if we show that

$$\int_{\Omega_i}[F_i(x) - F_R]dx \leqq c|R|\,\|f\|_{\mathrm{BMO}(Q)} \qquad (i = 1, 2)\,. \tag{4.6}$$

Consider first the case $i = 1$. Since $f_{\tilde R} \leqq F(x)$ for all $x$ in $R$, then certainly $f_{\tilde R} \leqq F_R$ so we may construct the Calderón-Zygmund decomposition [9, p. 17] for $f$ and $\tilde R$ with respect to the constant $F_R$. If the resulting sequence of pairwise disjoint cubes is denoted by $\{R_k\}_{k=1}^\infty$, and if $\bar R_k$ denotes the "parent" cube of $R_k$, then the following properties hold:

(i) $\bigcup_k \mathbf{R}_k \subset \tilde R$;

(ii) $f_{\bar R_k} \leqq F_R < f_{R_k}$ $\qquad (k = 1, 2, \cdots)$;

(iii) $|\bar R_k| = 2^n |R_k|$ $\qquad (k = 1, 2, \cdots)$;

(iv) $f \leqq F_R$ almost everywhere on $E = \tilde R\backslash(\bigcup_k R_k)$.

Define functions $b$ and $g$ on $Q$ by

$$b = \sum_k (f - f_{\bar R_k})\chi_{R_k}\,, \qquad g = \sum_k f_{\bar R_k}\chi_{R_k} + f\chi_E$$

so $f\chi_{\tilde R} = b + g$. It follows from (ii) and (iv) that

$$\|g\|_{L^\infty(Q)} \leqq F_R\,, \tag{4.7}$$

while on the other hand the John-Nirenberg lemma and (i) and (iii) give

$$\begin{aligned}\|b\|_{L^2(Q)} &= \Big\{\sum_k \int_{R_k}|f - f_{\bar R_k}|^2 dx\Big\}^{1/2} \leqq \Big\{\sum_k |\bar R_k|\frac{1}{|\bar R_k|}\int_{\bar R_k}|f - f_{\bar R_k}|^2 dx\Big\}^{1/2} \\ &\leqq c\Big(\sum_k 2^n|R_k|\Big)^{1/2}\|f\|_{\mathrm{BMO}(Q)} \leqq c|R|^{1/2}\|f\|_{\mathrm{BMO}(Q)}\,.\end{aligned} \tag{4.8}$$

Now it follows from the definition of $F_1$ that

$$F_1 \leqq M_Q(f\chi_{\tilde R}) = M_Q(b + g) \leqq M_Q b + M_Q g\,,$$

so applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\int_{\Omega_1} F_1(x)dx &\leqq |\Omega_1|^{1/2}\|M_Q b\|_{L^2(Q)} + |\Omega_1|\,\|M_Q g\|_{L^\infty(Q)} \\ &\leqq c|R|^{1/2}\|b\|_{L^2(Q)} + |\Omega_1|\,\|g\|_{L^\infty(Q)}\,.\end{aligned}$$

Combining this with (4.7) and (4.8), and subtracting $|\Omega_1|\,F_R$ from each side, we obtain (4.6) for $i = 1$.

The remaining case $i = 2$ will follow directly from the inequality

$$F_2(x) - F_R \leqq c\|f\|_{\mathrm{BMO}(Q)} \qquad (x \in \Omega_2) \tag{4.9}$$

which we now prove. Fix $x$ in $\Omega_2$ and let $P$ be any subcube of $Q$ that contains $x$ and has nonempty intersection with $Q \backslash \tilde{R}$. Clearly $|P| \geqq |R|$. Let $P'$ be the smallest subcube of $Q$ containing both $P$ and $R$. Then $|P'| \leqq 2^n|P|$. Arguing as before, we note that $f_{P'} \leqq F_R$. Hence

$$f_P - F_R \leqq f_P - f_{P'} \leqq \frac{1}{|P|}\int_P |f(y) - f_{P'}|dy \leqq 2^n\|f\|_{\mathrm{BMO}(Q)} ,$$

so taking the supremum over all such cubes $P$ we obtain (4.9). This establishes part (a).

The maximal function $F$ in the preceding proof is necessarily integrable over every cube $R$ (contained in $Q$) but this need not be the case when we extend to $\mathbf{R}^n$. However, if $f$ belongs to $\mathrm{BMO}(\mathbf{R}^n)$ and $R$ is any cube in $\mathbf{R}^n$, we can split the maximal function $F = Mf$ into the two parts analogous to $F_1$ and $F_2$ in the proof above and estimate these separately. The function $F_1$ is essentially a maximal function relative to a fixed cube and so may be estimated in terms of the BMO-norm of $f$ exactly as in the proof above. The function $F_2$ on the other hand is a supremum of averages of $f$ over "large" cubes which, by means of a fixed dilation, may be taken to contain $R$. But then each of these averages is bounded above by the maximal function $Mf$ evaluated at any point of $R$, so $F_2$ is bounded by $\inf_R Mf$. Hence we arrive at the following estimate

$$\frac{1}{|R|}\int_R (Mf)(x)dx \leqq c\big(\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} + \inf_{x \in R}(Mf)(x)\big) .$$

Since $R$ is arbitrary, it follows that the maximal function $F = Mf$ of a function $f$ in $\mathrm{BMO}(\mathbf{R}^n)$ is either identically infinite or else it is locally integrable (hence finite a.e. on $\mathbf{R}^n$). In the latter case, having established that the mean $F_R$ is finite, we may proceed exactly as in the proof of part (a) to show that $F$ is in $\mathrm{BMO}(\mathbf{R}^n)$. We omit the details.

UNIVERSITY OF SOUTH CAROLINA, COLUMBIA

## REFERENCES

[1] C. BENNETT and R. SHARPLEY, Weak-type inequalities for $H^p$ and BMO, Proc. Sympos. Pure Math. **35** (I) (1979), 201–229.

[2] P. L. BUTZER and H. BERENS, *Semi-Groups of Operators and Approximation*, Springer Verlag, New York, 1967.

[3] A. P. CALDERÓN, Spaces between $L^1$ and $L^\infty$ and the theorem of Marcinkiewiez, Studia Math. **26** (1966), 273–299.

[4] C. FEFFERMAN and E. M. STEIN, $H^p$ spaces of several variables, Acta Math. **129** (1972), 137–193.
[5] C. HERZ, A best-possible Marcinkiewicz theorem with applications to martingales, preprint.
[6] N. M. RIVIÉRE, Interpolacion a la Marcinkiewicz, Revìsta Unión Mat. Argentina **25** (1971), 363–377.
[7] S. SPANNE, Sur l'interpolation entre les espaces $\mathfrak{L}_k^{p\Phi}$, Ann. Scuola Norm. Sup. Pisa **20** (1966), 625–648.
[8] E. M. STEIN, Singular integrals, harmonic functions and differentiability properties of functions of several variables, Proc. Sympos. Pure Math. **10** (1967), 316–335.
[9] ———, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
[10] E. M. STEIN and G. WEISS, *Introduction to Harmonic Analysis on Euclidean Spaces*, Princeton University Press, 1971.
[11] E. M. STEIN and A. ZYGMUND, Boundedness of translation-invariant operators on Hölder and $L^p$ spaces, Ann. of Math. **85** (1967), 337–349.

(Received June 12, 1980)