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WEAK-TYPE INEQUALITIES IN ANALYSIS

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The purpose of this article is to survey some recent extensions of the Marcinkiewicz interpolation theorem due to C. Bennett - K. Rudnick and R.A. DeVore - S.D. Riemenschneider - R.C. Sharpley. These results are based on the observation that existing definitions of weak-type operators are somewhat inadequate and that a more natural definition is obtained by considering those operators that are dominated by the Calderón operator S_σ. Since the corresponding interpolation theorems can be lifted into a general Banach space context, this approach has important applications not only in harmonic analysis but in other areas such as approximation theory as well.

1. The Marcinkiewicz interpolation theorem

Denote by f^* the decreasing rearrangement on the interval $(0,\infty)$ of a measurable function f defined on some measure space (X,μ). For $0 < p,q \leq \infty$, the Lorentz space L^{pq} consists of all (classes of) measurable f on X for which the quasinorm

$$(1.1)\qquad ||f||_{p,q} = \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^q \, dt/t\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty, \end{cases}$$

is finite. The spaces L^{pq} increase with q; in particular

$$(1.2)\qquad L^{p1} \subseteq L^{pp} = L^p \subseteq L^{p\infty}, \qquad 1 \leq p \leq \infty.$$

A.P. Calderón's formulation [3] of the Marcinkiewicz interpolation theorem in terms of Lorentz spaces is as follows.

THEOREM 1.1. <u>Suppose</u>

$$(1.3)\qquad 0 < p_1 < p_2 \leq \infty; \qquad 0 < q_1, q_2 \leq \infty, \qquad q_1 \neq q_2.$$

<u>Let T be a quasilinear operator of weak types</u> (p_1,q_1) <u>and</u> (p_2,q_2). <u>Suppose</u>

$0 < \theta < 1$ *and let*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then T *is a bounded linear operator from* L^{pa} *into* L^{qa}, *i.e.*,

$$T : L^{pa} \to L^{qa}, \tag{1.4}$$

whenever $0 < a \leq \infty$.

Here, an operator is of weak type (p,q) if it is bounded from L^{p1} into $L^{q\infty}$. The case $p = \infty$ is excluded however because, as is clear from (1.1), the space $L^{\infty 1}$ contains only the zero-function. It is customary in this case to substitute L^{∞} for $L^{\infty 1}$ in the above definition; in particular, an operator is of weak type (∞,∞) if and only if it is bounded from L^{∞} into itself. This makeshift definition is satisfactory for some purposes (the Hardy-Littlewood maximal operator is then of weak type (∞,∞)) but is unsuitable for others (the Hilbert transform fails to be of weak type (∞,∞)).

In the next section we shall remedy this defect by modifying the definition of "weak type". As it turns out, most of the necessary machinery is already available: it has only to be assembled in a slightly different way.

2. The Calderón operator

With parameters as in (1.3), let ν be the slope of the line segment σ, say, joining the points $(1/p_i, 1/q_i)$, $i = 1,2$, in the plane.

The integral operator S_σ, defined by

$$(S_\sigma g)(t) = t^{-1/q_1} \int_0^{t^\nu} s^{1/p_1} g(s)\,ds/s + t^{-1/q_2} \int_{t^\nu}^{\infty} s^{1/p_2} g(s)\,ds/s, \qquad t > 0, \tag{2.1}$$

was introduced by A.P. Calderón [3] in connection with the following fundamental result for weak-type operators.

THEOREM 2.1 *(A.P. Calderón). Suppose, in addition to* (1.3), *that* $p_2 < \infty$. *Then a quasilinear operator* T *is of weak types* (p_1,q_1) *and* (p_2,q_2) *if and only if, for all* f *and some constant* c *independent of* f,

$$(Tf)^*(t) \leq c\, S_\sigma(f^*)(t), \qquad 0 < t < \infty. \tag{2.2}$$

The theorem characterizes the weak-type operators as those that are dominated by S_σ. It fails when $p_2 = \infty$ precisely because of the unnatural definition of "weak type (p_2,q_2)" in this case. Observe however that S_σ is perfectly well-defined when $p_2 = \infty$, and hence it makes sense to use (2.2) as a definition of "weak type", now for all values of the parameters.

DEFINITION 2.2 [1,2]. Suppose p_i, q_i, $i = 1,2$, satisfy (1.3). A quasilinear operator T is said to be of weak type $(p_1,q_1;\ p_2,q_2)$ if (2.2) holds.

Interpolation theorems follow rather easily with this definition. Thus, Calderón's proof of Theorem 1.1 proceeds by applying the L^{qa}-norm to both sides of (2.2); the classical Hardy inequalities then reduce the right-hand side to the L^{pa}-norm of f, hence establishing (1.4).

Our objective was to establish a definition of "weak type" that would apply to such operators as the Hilbert transform H. The desired result is that H be of weak type $(1,1;\ \infty,\infty)$:

$$(2.3) \qquad (Hf)^*(t) \le c\left(t^{-1}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\,ds/s\right), \qquad 0 < t < \infty.$$

This inequality can be derived fairly easily from a closely-related inequality due to R. O'Neil - G. Weiss [8]. Similarly, the maximal Hilbert transform and certain Calderón-Zygmund singular integrals are also of weak type $(1,1;\ \infty,\infty)$; complete details are given in [2].

3. Lorentz-Zygmund spaces

Besides the L^{pq} spaces, the Zygmund spaces $L^p(\log L)^\alpha$ also play an essential role in classical harmonic analysis. It is therefore desirable that both classes be incorporated into the interpolation scheme.

For $0 < p < \infty$, $-\infty < \alpha < \infty$, the Zygmund space $L^p(\log L)^\alpha$ consists of those f on the unit circle Γ for which

$$(3.1) \qquad \int_0^{2\pi}\left\{|f(e^{i\theta})|\ \log^\alpha(2 + |f(e^{i\theta})|)\right\}^p d\theta < \infty.$$

The Orlicz norms with which these spaces are usually endowed are of a quite different character to the Lorentz L^{pq} norms. However, it can be shown that (3.1) holds if and only if

$$(3.2) \qquad \left(\int_0^1\left\{f^*(t)(1-\log t)^\alpha\right\}^p dt\right)^{1/p} < \infty,$$

and this quantity is a Lorentz-type norm on $L^p(\log L)^\alpha$, equivalent to the Orlicz norm. Now it is easy to incorporate the Lorentz spaces and the Zygmund spaces in a larger class of spaces.

DEFINITION 3.1 [2]. *Suppose* $0 < p,q \leq \infty$ *and* $-\infty < \alpha < \infty$. *The Lorentz-Zygmund space* $L^{pq}(\log L)^{\alpha}$ *on* (X,μ) *consists of those (classes of) measurable functions* f *on* X *for which the quasinorm*

$$||f||_{p,q;\alpha} = \begin{cases} \left(\int_0^{\infty}\left\{t^{1/p}(1+|\log t|)^{\alpha}f^*(t)\right\}^q dt/t\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/p}(1+|\log t|)^{\alpha}f^*(t), & q = \infty, \end{cases}$$

is finite.

The spaces $L^{pq}(\log L)^{0}$ are immediately recognizable as the Lorentz spaces L^{pq}. On the unit circle, the spaces $L^{pp}(\log L)^{\alpha}$ are the Zygmund spaces $L^{p}(\log L)^{\alpha}$. Furthermore, the Zygmund spaces of functions whose $1/\alpha$-th powers are "exponentially integrable" occur as the Lorentz-Zygmund spaces $L^{\infty\infty}(\log L)^{-\alpha}$, and the O'Neil spaces $K^{p}(\log^{+} K)^{\alpha p}$ arise as the spaces $L^{p1}(\log L)^{\alpha}$. Several other types of spaces occurring in classical harmonic analysis can also be regarded as Lorentz-Zygmund spaces (cf. [2]).

4. Two interpolation theorems

Now we can present the interpolation theorems established by C. Bennett - K. Rudnick [2]. They describe the action of operators of weak type $(p_1,q_1; p_2,q_2)$ on the Lorentz-Zygmund spaces $L^{pa}(\log L)^{\alpha}$.

THEOREM 4.1. *Suppose* $0 < p_1 < p_2 \leq \infty$ *and* $0 < q_1,q_2 \leq \infty$, *with* $q_1 \neq q_2$. *Let* T *be a quasilinear operator of weak type* $(p_1,q_1; p_2,q_2)$. *Suppose* $0 < \theta < 1$ *and let*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then

$$T : L^{pa}(\log L)^{\alpha} \to L^{qa}(\log L)^{\alpha}$$

whenever $0 < a \leq \infty$ *and* $-\infty < \alpha < \infty$.

This result is a direct extension of Calderón's theorem (Theorem 1.1). The next result corresponds to the limiting cases $\theta = 0$ and $\theta = 1$ of Theorem 4.1. The principal feature is that the "index" $\alpha + 1/a$ of $L^{pa}(\log L)^{\alpha}$ always decreases by a factor of one. We state the theorem only in the form valid for finite measure spaces; the general statement can be found in [2, Theorem C].

THEOREM 4.2. Suppose $0 < p_1 < p_2 \le \infty$ and $0 < q_1 < q_2 \le \infty$. Let T be a quasi-linear operator of weak type $(p_1,q_1;\ p_2,q_2)$. Suppose $1 \le a \le b \le \infty$ and $-\infty < \alpha,\beta < \infty$. Then

(a) $$T : L^{p_1 a}(\log L)^{\alpha+1} \to L^{q_1 b}(\log L)^{\beta}$$

whenever $\alpha + 1/a = \beta + 1/b > 0$;

(b) $$T : L^{p_2 a}(\log L)^{\alpha+1} \to L^{q_2 b}(\log L)^{\beta}$$

whenever $\alpha + 1/a = \beta + 1/b < 0$.

These interpolation theorems contain (and in many cases extend) a wide variety of estimates from classical harmonic analysis, involving such operators as the Fourier transform, the Hardy-Littlewood maximal operator, the fractional integrals, and the Hilbert transform (cf. [2] for details). As an illustration we consider the classical estimates for the Weyl fractional-integral operators I_λ $(0 < \lambda < 1)$ on the unit circle (cf. [2, 12; Chapter XII]).

(a) (Hardy-Littlewood) $I_\lambda : L^p \to L^q$, $\quad 1 < p < q < \infty$, $\quad 1/p - 1/q = \lambda$;

(b) (Zygmund) $I_\lambda : L(\log L)^{1-\lambda} \to L^{(1-\lambda)^{-1}}$;

(c) (O'Neil) $I_\lambda : L(\log L)^{\alpha} \to K^{(1-\lambda)^{-1}}(\log^+ K)^{(1-\lambda)^{-1}(\alpha-1)}$, $\quad \alpha \ge 1$;

(d) (O'Neil) $I_\lambda : L(\log L)^{\alpha} \to L^{(1-\lambda)^{-1},1/\alpha}$, $\quad 0 < \alpha < 1$;

(e) (Zygmund) If $||f||_{\lambda^{-1}} \le 1$, there are constants γ and C independent of f such that $\displaystyle\int_{-\pi}^{\pi} \exp\left\{\gamma |I_\lambda f|^{\frac{1}{1-\lambda}}\right\} \le C < \infty$.

These diverse estimates can be neatly reformulated in terms of Lorentz-Zygmund spaces as follows:

(a') $I_\lambda : L^{p,p}(\log L)^{0} \to L^{q,q}(\log L)^{0}$, $\quad 1 < p < q < \infty$, $\quad 1/p - 1/q = \lambda$;

(b') $I_\lambda : L^{1,1}(\log L)^{1-\lambda} \to L^{(1-\lambda)^{-1},(1-\lambda)^{-1}}(\log L)^{0}$;

(c') $I_\lambda : L^{1,1}(\log L)^{\alpha} \to L^{(1-\lambda)^{-1},1}(\log L)^{\alpha-1}$, $\quad \alpha \ge 1$;

(d') $I_\lambda : L^{1,1}(\log L)^{\alpha} \to L^{(1-\lambda)^{-1},\ \alpha^{-1}}(\log L)^{0}$, $\quad 0 < \alpha < 1$;

(e') $I_\lambda : L^{\lambda^{-1},\lambda^{-1}}(\log L)^0 \to L^{\infty,\infty}(\log L)^{\lambda-1}$.

Since I_λ is of weak type $(1,(1-\lambda)^{-1};\ \lambda^{-1},\infty)$ it is a simple matter to check that (a') follows immediately from Theorem 4.1, (b'), (c') and (d') from the first part of Theorem 4.2, and (e') from the second part of Theorem 4.2.

5. Multilinear interpolation

Observe that the Calderón operator in (2.1) can be written as a kernel operator

$$(S_\sigma g)(t) = \int_0^\infty g(s)\, \Psi_\sigma(s,t)\, ds/s, \tag{5.1}$$

with kernel $\Psi_\sigma(s,t) = \min\left\{ s^{1/p_i} t^{-1/q_i} : i = 1,2, \quad 0 < s,t < \infty\right\}$. R.C. Sharpley [9] has shown that a natural generalization of this operator can be used to control multilinear interpolation. Such theorems are needed to deal with, for example, convolution operators [6] and tensor product operators [7].

Convolution operators C satisfy three initial estimates

$$C : \begin{cases} L^1 \times L^1 \to L^1 \\ L^1 \times L^\infty \to L^\infty \\ L^\infty \times L^\infty \to L^\infty \end{cases} \tag{5.2}$$

and, similarly, tensor product operators T satisfy two initial estimates

$$T : \begin{cases} L^1 \times L^1 \to L^1 \\ L^\infty \times L^\infty \to L^\infty \end{cases} \tag{5.3}$$

Thus, Sharpley [9] considers bilinear (or multilinear) operators satisfying m weak-type estimates, of the form

$$T(\chi_E,\chi_F)^{**}(t) \le c\, \frac{\mu(E)^{1/p_i}\, \mu(F)^{1/q_i}}{t^{1/u_i}}, \qquad i = 1,2,\ldots,m \tag{5.4}$$

(where $h^{**}(t) = t^{-1}\int_0^t h^*(s)ds$). Operators of this kind are said to satisfy the $\sigma((p_1,q_1;\ u_1),\ldots,(p_m,q_m;\ u_m))$ weak-type conditions. This leads easily to the inequality

$$T(\chi_E,\chi_F)^{**}(t) \le c\, \mathfrak{T}_\sigma(\mu(E),\mu(F);t)$$
$$\le \int_0^\infty \int_0^\infty \chi_E^*(r)\, \chi_F^*(s)\, \mathfrak{T}_\sigma(r,s;t) \frac{dr}{r}\frac{ds}{s},$$

since the kernel

$$\mathfrak{T}_\sigma(r,s;t) = \min\left\{r^{1/p_i} s^{1/q_i} t^{-1/u_i} : i = 1,2,\ldots,m\right\}$$

is concave in r and s, and decreasing in t. Passing to simple functions, we obtain the σ-weak-type inequality (cf. Definition 2.2)

(5.5)
$$T(f,g)^{**}(t) \le c\int_0^\infty \int_0^\infty f^*(r)\, g^*(s)\, \mathfrak{T}_\sigma(r,s;t) \frac{dr}{r}\frac{ds}{s}$$
$$\equiv c\, S_\sigma(f^*,g^*)(t).$$

As before, interpolation theorems are established by applying appropriate norms to both sides of (5.5). As corollaries, they yield the following results for convolution operators, which are of weak type $\sigma((1,1,1), (1,\infty,\infty); (\infty,\infty,\infty))$, and tensor product operators, which are of weak type $\sigma((1,1,1); (\infty,\infty;\infty))$ (cf. (5.2) and (5.3)), acting on Lorentz-Zygmund spaces.

THEOREM 5.1 [9]. Let C be a convolution operator. Then

(5.6)
$$C : L^{pa}(\log L)^\alpha \times L^{qb}(\log L)^\beta \to L^{uc}(\log L)^\gamma$$

where

$$1/p + 1/q = 1/u + 1, \qquad 1/a + 1/b \ge 1/c, \qquad \alpha + \beta \ge \gamma.$$

In fact, by considering estimates which are essentially inverse to (5.5), it is possible to obtain best-possible results for convolution (cf. [10]).

THEOREM 5.2 [9]. Let T be a tensor product operator. Then

$$T : L^{pa}(\log L)^\alpha \times L^{pb}(\log L)^\beta \to L^{pc}(\log L)^\gamma$$

where

$$1/a + 1/b \ge 1/c + 1, \qquad \gamma \le \min(\alpha,\beta).$$

These theorems contain as special cases many of the results established by O'Neil [6,7]. Finally, we remark that the S_σ-operator associated with the tensor products maps $L^1 \times L^1$ into $L^{1,\infty}$ and $L^2 \times L^2$ into $L^{2,\infty}$ but does not carry $L^p \times L^p$ into L^p (cf. [11]). This confirms a conjecture of J. Peetre to the effect that the Marcinkiewicz theorem does not extend to bilinear operators in this manner.

6. The conjugate-function operator

Let H be the conjugate-function operator (i.e., the Hilbert transform for the unit circle). There is the following inequality for H in terms of the modulus of continuity $\omega(f,t)$ (cf. [12, Chapter III, p.121]):

$$t^{-1}\omega(Hf,t) \leq c\left\{t^{-1}\int_0^t s^{-1}\omega(f,s)ds + \int_t^\infty s^{-1}\omega(f,s)ds/s\right\}. \tag{6.1}$$

Clearly, if the quantities $t^{-1}\omega(h,t)$ are replaced by decreasing rearrangements h^*, then the fundamental inequality (2.3) results. This observation was made by R.A. DeVore - S.D. Riemenschneider - R.C. Sharpley [4], and it suggested to them that Marcinkiewicz-type theorems might play a role far outside of the familiar Banach function space setting. This prompted their development of a weak-type interpolation theory in a general Banach space context. As we shall see below and in section 8, it has a number of interesting applications.

Notice that once a "weak-type" inequality such as (6.1) (or (2.3)) is established, then, by applying Lorentz-Zygmund norms to each side, we obtain interpolation theorems that are entirely analogous to Theorems 4.1 and 4.2.

Thus, in the present case we obtain the estimates

$$H : B_\infty^{\theta,a;\alpha} \to B_\infty^{\theta,a;\alpha}, \qquad 0 < \theta < 1, \qquad a > 0, \qquad -\infty < \alpha < \infty, \tag{6.2}$$

where the norm in $B_p^{\theta,a;\alpha}$ is given in terms of the L^p-modulus of continuity by

$$||f||_{B_p^{\theta,a;\alpha}} = \left|\left|\frac{\omega(f,.)_p}{(.)}\right|\right|_{q,a;\alpha}, \qquad q = (1-\theta)^{-1}. \tag{6.3}$$

These spaces reduce to the Besov spaces when $\alpha = 0$, and to the Lipschitz spaces when we further require $a = p = \infty$. Thus (6.2) contains, for instance, the familiar result on the boundedness of H on Lip(θ), for $0 < \theta < 1$. Similarly, in the limiting cases $\theta = 0$ and $\theta = 1$ (corresponding to Theorem 4.2) we obtain as easy corollaries the following well-known classical estimates:

$$\omega(f,t) = O(t) \quad\Rightarrow\quad \omega(Hf,t) = O(t|\log t|);$$

$$\omega(f,t) = O(|\log t|^{-(\alpha+1)}) \Rightarrow \omega(Hf,t) = O(|\log t|^{-\alpha}), \quad \alpha > 0.$$

For further details, and extensions to higher order moduli, see [4].

7. The general theory

The crucial link between the inequalities (2.3) and (6.1), and the means by which we can pass to a more general setting, is of course the Peetre K-functional $K(f,t)$ (cf. [4]). Thus, for the pair (L^1,L^∞) the K-functional is given by $K(f,t;\, L^1,L^\infty) = \int_0^t f^*(s)ds$ (so its derivative $k(f,t)$ coincides with $f^*(t)$), and for the pair (C,Lip 1) it is given (up to equivalence) by $K(f,t;\, C^1,\text{Lip } 1) \approx \omega(f,t)$.

It is therefore natural to consider inequalities of the form

$$\text{(7.1)} \qquad t^{-1}K(Tf,t;\, Y_1,Y_2) \le c\; S_\sigma((.)^{-1}K(f,(.);\, X_1,X_2)(t), \qquad t > 0,$$

for appropriate segments σ. An operator T satisfying (7.1) is said to be of generalized weak-type σ with respect to the compatible couples (X_1,X_2) and (Y_1,Y_2) of Banach spaces (cf. [4]).

In many instances only a part of the Calderón operator is needed. Thus, if $S_{\sigma,1}(g)$ and $S_{\sigma,2}(g)$ denote respectively the first and second integrals in (2.1), we extend the definition of S_σ to mean

$$S_\sigma = \begin{cases} S_{\sigma,1} & \text{if } \sigma = \sigma(p_1,q_1;\, p_2,q_2]; \\ S_{\sigma,2} & \text{if } \sigma = \sigma[p_1,q_1;\, p_2,q_2); \\ S_{\sigma,1} + S_{\sigma,2} & \text{if } \sigma = \sigma(p_1,q_1;\, p_2,q_2), \end{cases}$$

where the closed bracket indicates that the corresponding endpoint is included in the segment σ.

With this notation, the inequality (6.1) asserts that H is of generalized weak type $\sigma(1,1;\,\infty,\infty)$ for the pair C and Lip 1. Similarly, if the functional $t^{-1}K(f,t)$ is replaced by $k(f,t)$, the inequality (2.3) shows that H is of generalized weak type $\sigma(1,1;\,\infty,\infty)$ for the pair L^1 and L^∞.

Notice that it is the presence of the S_σ operator in (7.1) which enables us to establish interpolation theorems exactly analogous to Theorems 4.1 and 4.2. This process can be carried out regardless of the particular choice of functional (such as $t^{-1}K(f;t)$ or $k(f,t)$, as above, or the "degree of approximation functional" $t^{-1}E(f,t)$, as in the next section). Different types of functionals do, of course, yield different types of spaces appearing in those interpolation theorems, and so the choice is made to suit the particular application in mind. This freedom of choice provides the theory with great flexibility.

8. Applications

Marchaud's inequality [5] for the circle

$$(8.1)\qquad \frac{\omega_r(f,t)_p}{t^r} \le c \int_t^\infty \frac{\omega_{r+k}(f,s)_p}{s^r}\frac{ds}{s}$$

can be rewritten to show that the identity operator is of generalized weak type $\sigma[1,1; (r+k)/k,\infty)$ with respect to (L^p, W_p^{r+k}) and (L^p, W_p^r), where $\omega_r(f,t^{1/r})_p \sim K(f,t;L^p,W_p^r)$ and W_p^r is the Sobolev space of L^p-functions whose r-th order distributional derivatives are in L^p. The basic inequality (8.1) is the converse of the trivial estimate

$$(8.2)\qquad \omega_{r+k}(f,s)_p \le c\, \omega_r(f,s)_p.$$

Consider, for simplicity, the case $r = k = 1$. Using (8.1) and (8.2), we find that when $0 \le \theta < 1$ it is possible to replace $\omega(f,t)_p$ in (6.3) by the second order modulus $\omega_2(f,t)_p$ to get an equivalent norm. In the extreme case $\theta = 1$, $\alpha = -1$ and $a = \infty$ there is the anticipated loss of logarithm:

$$\omega_2(f,t)_p = O(t) \Rightarrow \omega(f,t)_p = O(t|\log t|).$$

Inequalities similar to (8.1) and (8.2) hold for moduli of smoothness for domains in R^n satisfying a cone condition, or for equibounded C_o-semigroups of operators (cf. [4, §5]). The reduction theorems for Besov spaces follow from the inequality

$$\omega_{r+k}(f,t)_p \le c\, t^k \sup_{|\beta|=k} \omega_r(D^\beta f,t)_p$$

and the weak-type inequality for differentiation

$$\omega_r(D^\beta f,t)_p \le c \int_0^t \frac{\omega_{r+k}(f,s)_p}{s^k}\frac{ds}{s},$$

where β is a multi-index of length k. For more details and the reduction theorems in terms of semigroup generators see [4, §5].

Jackson's direct approximation theorem on the circle [5] gives

$$E(f,1/t)_p \le c\omega(f,t)_p, \qquad 0 < t \le 1,$$

where $E(f,s)_p$ is the error of approximation in L^p by trigonometric polynomials of degree at most [s]. Bernstein's inverse theorem [5] can be interpreted as the weak-type inequality

$$t^{-1}\omega(f,t)_p \le c\, S_\sigma(E(f,.)_p)(t),$$

where $\sigma = \sigma(1,\infty;\ \infty,1]$. Consequently, we obtain the standard results

$$\omega(f,t)_p = O(t^\theta) \Rightarrow E(f,1/t) = O(t^\theta), \qquad 0 \le \theta < 1.$$

The loss of logarithm at $\theta = 1$ occurs as before [4, §8].

Next, we consider embeddings of Besov spaces. If Ω is a bounded domain satisfying a cone property, or all of R^n, there is an estimate of the form

$$\omega_r(f,t)_q \le c \int_0^t \omega_r(f,s)_p\ s^{-\theta}\ ds/s, \tag{8.3}$$

where $r \ge n$, $1 \le p \le q \le \infty$, and $\theta = n/p - n/q$. Hence the identity operator is of generalized weak type $\sigma(r/(r-\theta),1;\ \infty,r/\theta]$ for the pairs (L^p,W^r_p) and (L^q,W^r_q). Interpolation gives the embeddings

$$B_p^{\lambda+\theta,a;\alpha} \subset B_q^{\lambda,a;\alpha}, \qquad \lambda > 0,$$

and

$$B_p^{\theta,1} \subset L^q,$$

with similar results involving the additional log term at the endpoints.

In order to estimate growth in terms of smoothness, the inequality

$$f^{**}(t) \le c \int_{t^{1/n}}^{\infty} \omega_n(f,s)_p\ s^{-n/p}\ ds/s, \qquad 1 < p < \infty, \tag{8.4}$$

is established in [4, §6]. This asserts that the identity operator is of generalized weak type $\sigma[1,p;\ p',\infty)$ with respect to (L^p,W^n_p) and (L^1,L^∞). By interpolation we obtain the embeddings

$$B_p^{\theta,a;\alpha} \subset L^{q,a;\alpha}, \qquad B_p^{n/p,1} \subset L^\infty,$$

where $p < q$ and $\theta = n/p - n/q$, etc.

As a final application we mention absolute convergence of Fourier transforms $\hat{f}$ on R^n. In [4, §8] it is shown that

$$(\hat{f})^{**}(t) \le ct^{-1} \int_{t^{-1/n}}^{\infty} \omega_r(f,s)_p\ s^{-n/p}\ ds/s,$$

i.e., the Fourier transform is of generalized weak type $\sigma[1,p';\ pr/(pr-n),1)$ for the pairs (L^p,W^r_p) and (L^1,L^∞). Interpolation gives

$$(B_p^{\theta,a;\alpha})^\wedge \subset L^{q,a;\alpha}\ ; \qquad (B_p^{n/p,1})^\wedge \subset L^1, \text{ etc.}$$

Weak-type methods also play an important role in the development of abstract interpolation itself. Indeed, the equivalence, reiteration, and multilinear theorems of the Peetre J-and K-methods can be written as weak-type inequalities (cf. [4, §§4,9]).

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