CHARACTERIZATION OF INTERMEDIATE SPACES OF M_ϕ SPACES

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A natural problem in interpolation theory is to find necessary and sufficient conditions for interpolation properties to hold among spaces of the same type. This paper serves to provide a solution for M_ϕ spaces which appear in fundamental roles in classical interpolation theorems such as the Marcinkiewicz and Stein-Weiss theorems ([1],[2],[10]) as well as in the class of rearrangement invariant Banach function spaces ([6],[8]).

A partial treatment of this problem was presented in [7] where the two cases of order-continuous operators on M_ϕ and arbitrary operators on M_ϕ^0 (the closure of the simple functions in M_ϕ) were solved. In both cases the method of proof was a duality argument applied to a result of Lorentz and Shimogaki [4] on Λ spaces. Here a straightforward proof is given for M_ϕ spaces from which these results follow as corollaries.

If ϕ is a positive, decreasing, locally integrable function on a possibly infinite interval $(0,\ell)$ and $\Phi(t) = \int_0^t \phi(s)\,ds$, then M_ϕ is defined in the literature as the linear space of measurable functions on $(0,\ell)$ which satisfy

$$\| f \| = \sup_t \{f^{**}(t)\ t/\Phi(t)\} < \infty$$

where $f^{**}(t) = \int_0^t f^*(s)ds$ and f^* is the decreasing rearrangement of $|f|$. For convenience we wish to use a slightly different but totally compatible definition more in line with the theory as presented in [8].

DEFINITION. *The space* $M(\phi)$ *is the linear space of measurable functions for which the norm*

$$\| f \|_{\phi} = \sup_t \{f^{**}(t)\ \phi(t)\}$$

is finite.

The correspondence between these two definitions is $M(\phi)=M_{\psi}$ and $M(\psi)=M_{\phi}$ with equivalent norms where ψ is the derivative of $t/\Phi(t)$.

A pair of spaces (X,Y) is called an interpolation pair for $[(X_1,Y_1),(X_2,Y_2)]$ if each operator which is bounded from X_i to Y_i $(i=1,2)$ has a unique extension to a bounded operator from X to Y.

THEOREM. *A necessary and sufficient condition that* $(M(\phi),M(\psi))$ *be an interpolation pair for* $[(M(\phi_1),M(\psi_1)),(M(\phi_2),M(\psi_2))]$ *is that the condition*

$$(*) \qquad \Psi(t) \min_{i=1,2} \{\Phi_i(s)/\Psi_i(t)\}/\Phi(s) \leqslant C$$

hold for all s *and* t *in* $(0,\ell)$ *where* $\Psi(t)=\int_0^t \psi(r)dr$ *and* $\Phi(s)=\int_0^s \phi(r)dr$.

Before proving this theorem we need to establish several lemmas.

LEMMA 1. <u>Suppose that g is a positive, decreasing function belonging to a M(ϕ) space which is constant on each component of an open set A, then</u>

$$\|g\|_\phi \leqslant 2 \sup \{g^{**}(s)\Phi(s) \mid s \notin A\}.$$

PROOF. Suppose (a,b) is a component of A so that $g(a^+)$ is the value of g on (a,b). Letting $I=\int_0^a g(s)ds$, we have for each r in (a,b)

$$\begin{aligned} g^{**}(r)\Phi(r) &= \{I+(r-a)g(a^+)\}\Phi(r)/r \\ &\leqslant \{I-ag(a^+)\}\Phi(r)/r + g(a^+)\Phi(r). \end{aligned}$$

But g is a decreasing function, so $I-ag(a^+)$ is nonnegative. This together with the facts that $\Phi(r)$ increases while $\Phi(r)/r$ decreases implies

$$\begin{aligned} g^{**}(r)\Phi(r) &\leqslant \{I-ag(a^+)\}\Phi(a)/a + g(a^+)\Phi(b) \\ &\leqslant g^{**}(a)\ \Phi(a) + g^{**}(b)\Phi(b) \\ &\leqslant 2 \sup \{g^{**}(s)\Phi(s) \mid s \notin A\}. \end{aligned}$$

Applying the supremum on the left over all members of A as well as its complement, we obtain the desired inequality.

For each t in $(0,\ell)$ define the set $A_t=\{s \mid \Phi_1(s)/\Psi_1(t) < \Phi_2(s)/\Psi_2(t)\}$ and the set B_t to be the complement of the closure of A_t. Since the functions Φ_i are continuous, both these sets are open.

MAIN LEMMA. <u>Suppose condition (*) holds and g is a positive, decreasing function in M(ϕ), then g can be written as a sum $g=g_1+g_2$ where g_1(resp. g_2) is a positive decreasing function in M(ϕ_1) (resp. M(ϕ_2)) and is constant on each component of B_t (resp. A_t).</u>

PROOF. Write A_t as the disjoint union $\bigcup_j (a_j,b_j)$.

Define h_j by

$$h_j(s) = [\min(g(s),g(a_j^+))-g(b_j^-)]_+$$

and then let

$$g_1(s) = \sum_j h_j(s).$$

Since each h_j is constant on each component of B_t and is a positive decreasing function, g_1 has the same properties.

Now we let $g_2=g-g_1$ and notice that g_2 is constant on each component of A_t and is positive decreasing. Estimating with Lemma 1 and condition (*), we have

$$\begin{aligned} \|g_1\|_{\phi_1} &\leq 2\sup\{g_1^{**}(s)\phi_1(s) \mid s\notin B_t\} \\ &\leq 2C\sup\{g_1^{**}(s)\phi(s) \mid s\notin B_t\}\Psi_1(t)/\Psi(t) \\ &\leq \text{const.}\|g_1\|_\phi \\ &\leq \text{const.}\|g\|_\phi < \infty . \end{aligned} \tag{1}$$

Similarly,

$$\|g_2\|_{\phi_2} \leq \text{const.}\|g_2\|_\phi \leq \text{const.}\|g\|_\phi < \infty. \tag{2}$$

DEFINITION. $M^0(\phi)$ *is the closure in the norm of* $M(\phi)$ *of the simple functions with finite support. It is well known* [5] *that* $M^0(\phi)$ *consists of those functions in* $M(\phi)$ *with absolutely continuous norm, i.e.*

$$\lim_{\delta\to 0} \|f^* \chi_{E(\delta)}\|_\phi = 0 \tag{3}$$

where $E(\delta) = (0,\delta)\cup(1/\delta,\ell)$.

LEMMA 2. *If condition* (*) *holds, then* $M(\phi)$ *is continuously embedded in* $M(\phi_1)+M(\phi_2)$ *which is the Banach space of all functions* f *for which the norm*

$$\| f \|_{\phi_1 + \phi_2} = \inf \{ \| f_1 \|_{\phi_1} + \| f_2 \|_{\phi_2} \mid f = f_1 + f_2 \}$$

is finite. Moreover, $M^o(\phi)$ *is continuously embedded in* $M^o(\phi_1) + M^o(\phi_2)$.

PROOF. Suppose f belongs to $M(\phi)$, then the main lemma applied to the function $g = f^*$ produces functions g_1 and g_2. Applying the measure-preserving map $f^* \to f$ to each, we obtain functions f_1 and f_2 so that $f_1 + f_2 = f$ and $f_i^* = g_i$. But relations (1) and (2) along with the fact that each $M(\phi)$ space is rearrangement-invariant gives

$$\begin{aligned} \| f \|_{\phi_1 + \phi_2} &\leq \| f_1 \|_{\phi_1} + \| f_2 \|_{\phi_2} = \| g_1 \|_{\phi_1} + \| g_2 \|_{\phi_2} \\ &\leq 2 \text{ const. } \| g \|_{\phi} = 2 \text{ const. } \| f \|_{\phi} \end{aligned} \tag{4}$$

where the constant is independent of f.

For the second part of the lemma we note that f belongs to $M^o(\phi)$ is equivalent to $(f^*)_\delta$ converging to zero in $M(\phi)$ as $\delta \to 0$ where

$$(f^*)_\delta(s) = f^*(s) \, \chi_{E(\delta)}(s) + f^*(1/\delta) \, \chi_{(\delta, 1/\delta)}(s).$$

To see this we only need to show that

$$\| f^* (1/\delta) \, \chi_{(\delta, 1/\delta)} \|_\phi \to 0 \quad \text{as} \quad \delta \to 0$$

since

$$\| (f^*)_\delta \|_\phi \leq \| f^* \chi_{E(\delta)} \|_\phi + \| f^*(1/\delta) \, \chi_{(\delta, 1/\delta)} \|_\phi .$$

Letting $\Delta = 1/\delta$, we have

$$\| f^*(\Delta) \, \chi_{(1/\Delta, \Delta)} \|_\phi \leq f^*(\Delta) \, \Phi(\Delta)$$

$$\leq \; 2\, f^*(\Delta)\, \Phi(\Delta/2)$$

(5) $$= \; 2\, \| f^*(\Delta)\, \chi_{(\Delta/2,\Delta)} \|_\phi$$

$$\leq \; 2\, \| f^*\, \chi_{(\Delta/2,\ell)} \|_\phi$$

$$\leq \; 2\, \| f^*\, \chi_{E(2\delta)} \|_\phi$$

since Φ is concave and f^* decreases. But relation (3) states that as $\delta \to 0$ the right hand side of inequality (5) tends to zero thereby proving the claim.

Suppose now that f belongs to $M^o(\phi)$, then the first part of this lemma shows that $f = f_1 + f_2$ where f_1 belongs to $M(\phi_i)$, i=1,2. We show that $\| (f_i^*)_\delta \|_{\phi_i}$ tends to zero with δ which proves the second part of the lemma. But since $(g)_\delta$ is constant on the interval $(\delta, 1/\delta)$ it is easily seen that the Main Lemma gives the functions $(g_1)_\delta$ and $(g_2)_\delta$ as the decomposition for $(g)_\delta = (f^*)_\delta$ with inequalities (1) and (2) holding, i.e.

$$\| (g_i)_\delta \|_{\phi_i} \;\leq\; \text{const.}\; \| (g)_\delta \|_\phi \, .$$

Since the right hand side goes to zero with δ, we see that

$$\| (f_i)^*\, \chi_{E(\delta)} \|_{\phi_i} \;\to\; 0 \quad \text{as } \delta \to 0.$$

LEMMA 3. *If condition (*) holds, then $M(\psi_1) \cap M(\psi_2)$ is continuously embedded in $M(\psi)$ where the intersection is equipped with the norm*

$$\| f \|_{\psi_1 \cap \psi_2} \;=\; \max_{i=1,2} \left(\| f \|_{\psi_i} \right).$$

Furthermore, $M^o(\psi_1) \cap M^o(\psi_2)$ is continuously embedded in $M^o(\psi)$.

PROOF. Suppose f belongs to the intersection, then condition (*) with s=1 and taking the supremum on both sides over t gives

$$(6) \qquad \| f \|_{\psi} \leq C \max_{i=1,2} \{\phi(1)/\phi_i(1)\} \; \| f \|_{\psi_1 \cap \psi_2}.$$

The inequality (6) applied to the function $f^* \chi_{E(\delta)}$ proves the second statement in the lemma.

PROOF OF THE THEOREM. The necessity of the condition is well known [4] and can be easily seen by considering the operators $T_{s,t}[f](r) = (\int_0^s f(u)\, du / s)\, \chi_{(0,t)}(r)$ which has operator norm from $M(\phi)$ to $M(\psi)$ equal $\Psi(t)/\Phi(s)$. The condition

$$\| T_{s,t} \| \leq C \max_{i=1,2} \| T_{s,t} \|_i$$

is exactly condition (*) where $\| T \|$ (resp. $\| T \|_i$) is the operator norm of T from $M(\phi)$ to $M(\psi)$ (resp. $M(\phi_i)$ to $M(\psi_i)$).

To show the sufficiency of (*), suppose f belongs to $M(\phi)$ and that T is a bounded operator from $M(\phi_i)$ to $M(\psi_i)$, i=1,2. Let M be the maximum of $\| T \|_i$, i=1,2. Lemma 2 asserts that $M(\phi)$ is contained in the domain of T and hence Tf is defined and belongs to $M(\psi_1)+M(\psi_2)$. Let t be a fixed but arbitrary member of $(0,\ell)$. Using the Main Lemma applied to t and the function $g=f^*$, we have that $f^*=g_1+g_2$ where g_1 and g_2 are positive decreasing functions constant on each component of B_t and A_t, respectively. Applying the measure-preserving map $f^* \to f$ (see Lemma 2 of [1]), we obtain functions f_i such that $f_i^* = g_i$ and $f=f_1+f_2$. By Lemma 1 and the fact that $M(\phi_1)$ is a rearrangement invariant space, we have

$$\begin{aligned}(7)\qquad \| f_1\|_{\phi_1} = \| g_1\|_{\phi_1} &\leqslant 2 \sup\{g_1^{**}(s)\phi_1(s) \mid s\notin B_t\}\\ &\leqslant 2 \sup\{f^{**}(s)\phi_1(s) \mid s\notin B_t\}.\end{aligned}$$

Pick s_1 in the closure of A_t so that for $\varepsilon>0$

$$(8)\qquad \| f_1\|_{\phi_1} \leqslant (2+\varepsilon)\ f_1^{**}(s_1)\ \Phi_1(s_1).$$

Using condition (*) for s_1 and t and inequality (8), we obtain

$$\begin{aligned}(Tf_1)^{**}(t)\ \Psi(t) &\leqslant C \max_{i=1,2} [(Tf_1)^{**}(t)\ \Psi_i(t)/\Phi_i(s_1)]\Phi(s_1)\\ &= C\ [(Tf_1)^{**}(t)\ \Psi_1(t)/\Phi_1(s_1)]\Phi(s_1)\\ (9)\qquad &\leqslant M\ C\ (\ \| f_1\|_{\phi_1}\ /\Phi_1(s_1)\ )\ \Phi(s_1)\\ &\leqslant (2+\varepsilon)\ M\ C\ f_1^{**}(s_1)\ \Phi(s_1)\\ &\leqslant (2+\varepsilon)\ M\ C\ \| f_1\|_{\phi}\ .\end{aligned}$$

But t and ε were arbitrary, so

$$(10)\qquad \| Tf_1\|_{\psi} \leqslant 2\ M\ C\ \| f_1\|_{\phi}\ .$$

Similarly,

$$(11)\qquad \| Tf_2\|_{\psi} \leqslant 2\ M\ C\ \| f_2\|_{\phi}\ .$$

Inequalities (10) and (11) with the triangle inequality and the fact that $\| f_i\|_{\phi} \leqslant \| f\|_{\phi}$ give

$$\| Tf\|_{\psi} \leqslant 4\ M\ C\ \| f\|_{\phi}\ .$$

COROLLARY. _A necessary and sufficient condition that_ $(M^{O}(\phi),M^{O}(\psi))$ _be an interpolation pair for_ $[(M^{O}(\phi_1), M^{O}(\psi_1)),(M^{O}(\phi_2),M^{O}(\psi_2))]$ _is that condition_ (*) _hold_.

PROOF. As noted in the theorem, the necessity of (*) is well-known. For the sufficiency of the condition, let T be a bounded operator from $M^o(\phi_i)$ to $M^o(\psi_i)$, i=1,2, and let f belong to $M^o(\phi)$. By the second part of Lemma 2, Tf is defined and belongs to $M^o(\psi_1)+M^o(\psi_2)$. Since f belongs to $M^o(\phi)$, there is a sequence of simple functions $\{f_n\}$ converging to f in $M(\phi)$. Applying inequalities (9) and (11), we see that Tf_n converges to Tf in $M(\psi)$. Hence we only need to show that Tf_n belongs to $M^o(\psi)$. But f_n belongs to $M^o(\phi_1)\cap M^o(\phi_2)$, so Tf_n is in $M^o(\psi_1)\cap M^o(\psi_2)$. By the second part of Lemma 3, Tf_n belongs to $M^o(\psi)$.

REMARK 1. A class of spaces similar to $M(\phi)$ spaces are the Marcinkiewicz spaces $M^*(\phi)$ which consists of the measurable functions of finite quasi-norm

$$\| f \|_\phi^* = \sup_t f^*(t)\Phi(t).$$

In most cases (see [8]) $\| f \|_\phi^*$ is equivalent to the norm $\| f \|_\phi$. If we restrict ourselves to Marcinkiewicz spaces, we can also get that condition (*) is necessary and sufficient for interpolation. In fact, the basic ingredients of the proof are still the main lemma and inequality (9), and it even simplifies somewhat.

REMARK 2. Finally, it should be noted that the Lorentz-Shimogaki result mentioned before follows from the theorem by a simple duality argument and the fact that the Banach space dual of a Lorentz Λ_ϕ space is M_ϕ.

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