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Interpolation of n Pairs and Counterexamples Employing Indices

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1. INTRODUCTION

The purpose of this paper is to make two modest contributions to the theory of interpolation of operators on function spaces. The first deals with the modification of interpolation theorems to handle interpolation of n pairs. This work was partially begun in [8, 9] where strong type interpolation theorems were obtained using techniques easily altered to fit the n pair situation. Section 2 presents a weak type theory of interpolation of n pairs of spaces using the Calderón operator S_σ. In order to show that S_σ is a "maximal" weak type operator, the proof in [6] for $n = 2$ is simplified and extended.

Section 3 deals with counterexamples to interpolation theorems which make use of indices derived from the fundamental functions of the spaces in the interpolation scheme. We show that interpolation theorems of Semenov [5] and Zippin [11] are incorrect by using a space furnished by Shimogaki [10]. Under a suitable hypothesis a weak type theorem involving the fundamental indices does hold, but follows from a well-known result of Boyd [1] without employing the main ideas set forth in [5] and [3].

2. WEAK INTERPOLATION OF n PAIRS

A rearrangement invariant (symmetric in [5]) function space is a Banach space X of Lebesgue measurable functions on $(0, l)$ (possibly an infinite interval) such that the following conditions hold:

(i) if $|g| \leqslant |f|$ a.e. and $f \in X$, then $g \in X$ and $\|g\| \leqslant \|f\|$.

(ii) if $|g|$ is equimeasurable with $|f|$ and $f \in X$, then $g \in X$ and $\|g\| = \|f\|$.

(iii) if $mE < \infty$, then there exists $C_E > 0$ such that $\int_E f\,ds \leq C_E \|f\|$ for each $f \in X$.

(iv) if $mE < \infty$, then $\|\chi_E\| < \infty$ where χ_E is the characteristic function of the set E.

If, in addition, the norm satisfies the sequential Fatou property, i.e.,

(v) if $0 \leq f_n \uparrow f$ a.e. and $f_n \in X$ with $\|f_n\| \leq M$, then $f \in X$ and $\|f\| \leq M$,

then X is called a rearrangement invariant Banach function space [4]. The associate space X' consists of all measurable functions g on $(0, l)$ for which

$$\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \left| \int_0^l f(s)\, g(s)\, ds \right|,$$

is finite.

The fundamental function of a rearrangement invariant function space X is just $\varphi_X(t) = \|\chi_{(0,t)}\|_X$. We refer the reader to [11] and [6] for many of the properties of the functions φ_X. We use the fact that any rearrangement invariant function space X can be equivalently renormed so as to insure that φ_X is concave. We assume that all spaces in the remainder of this paper are renormed in this fashion.

An operator T is of weak type (X, Y) if it satisfies the inequality

$$\sup_t \{(Tf)^*(t)\, \varphi_Y(t)\} \leq M \int_0^l f^*\, d\varphi_X, \tag{1}$$

for all simple functions f of finite support, where f^* is the decreasing rearrangement of $|f|$. A pair of rearrangement invariant function spaces (X, Y) is weak intermediate for the interpolation scheme $\sigma = [(X_1, Y_1),..., (X_n, Y_n)]$ if each operator which is weak type (X_i, Y_i), $i = 1, 2,..., n$, has a unique extension to a bounded operator from X to Y. We assume throughout that $\min_{i=1,2,...,n}\{\varphi_{X_i}(0+)\} = 0$.

As in the literature [2, 6, 11] Calderón's operator can be defined for σ by

$$S_\sigma f(t) = \int_0^l f(s)\, \partial\Psi(s, t)/\partial s\, ds,$$

where $\Psi(s, t) = \min_{i=1,2,...,n}[\varphi_{X_i}(s)/\varphi_{Y_i}(t)]$.

The main result of this section then reads:

THEOREM. *A necessary and sufficient condition that a pair (X, Y) be weak intermediate for $\sigma = [(X_1, Y_1),..., (X_n, Y_n)]$ is that $S_\sigma f \in Y$ for each $f \in X$.*

We prove this theorem for $n = 3$ and refer the reader to [6] for those parts of the proof that carry over without change from the case $n = 2$. A deviation from this policy is a proof of the necessity of the theorem since the following simple argument does not appear in the literature.

LEMMA 1. *S_σ is of weak type (X_i, Y_i).*

Proof. We show that the lemma is true for $i = 1$. Suppose f is a simple function with finite support, then f^* can be written as

$$f^*(s) = \sum_{j=1}^{k} a_j \chi_{(0,s_j)}(s).$$

Since,

$$\int_0^l |hg| \, ds \leqslant \int_0^l h^* g^* \, ds,$$

for all measurable functions h and g, and since $\partial\Psi(s, t)/\partial s$ is a decreasing function of s for each t, we have

$$|S_\sigma f(t)| \leqslant S_\sigma[f^*](t). \tag{2}$$

But then inequality (2) gives

$$(S_\sigma f)^*(t) \leqslant S_\sigma[f^*](t),$$

since $S_\sigma[f^*]$ decreases. Estimating $S_\sigma[f^*]$ we obtain

$$\begin{aligned} S_\sigma[f^*](t) &= \sum_{j=1}^{k} a_j \Psi(s_j, t) \leqslant \sum_{j=1}^{k} a_j \varphi_{X_1}(s_j)/\varphi_{Y_1}(t) \\ &= \int_0^l f^*(s) \, d\varphi_{X_1}(s)/\varphi_{Y_1}(t) = \|f\|_{\Lambda(X_1)}/\varphi_{Y_1}(t), \end{aligned} \tag{3}$$

where $\|f\|_{\Lambda(X_1)} = \int_0^l f^* \, d\varphi_{X_1}$. In fact, using relations (2) and (3) with the monotone convergence theorem, we can obtain

$$(S_\sigma f)^*(t) \varphi_{Y_1}(t) \leqslant \|f\|_{\Lambda(X_1)}, \tag{4}$$

for all f for which the right-hand side is finite.

We shall need the following decomposition result for the sufficiency of the theorem (compare [6, Lemma 4.3]).

LEMMA 2. *Suppose $\mathscr{D}$ denotes the set of positive decreasing functions on*

$(0, l)$. If $g \in \mathscr{D}$, then for each t there exist functions $g^{(1)}, g^{(2)}, g^{(3)} \in \mathscr{D}$ such that $\sum_{i=1}^{3} g^{(i)} = g$ and

$$S_\sigma[g^{(i)}](t) = \| g^{(i)} \|_{\Lambda(X_i)} / \varphi_{Y_i}(t). \tag{5}$$

Proof. First assume $g \in \mathscr{D}$ is a step function with finite support and therefore can be written as

$$g(s) = \sum_{j=1}^{k} a_j \chi_{(0, s_j)}(s).$$

Define the index sets J_i, $i = 1, 2, 3$, by

$$J_i = \{1 \leqslant j \leqslant k \mid \varphi_{X_i}(s_j)/\varphi_{Y_i}(t) = \Psi(s_j, t)\} \Big\backslash \Big(\bigcup_{m=1}^{i-1} J_m \Big).$$

Notice that the disjoint union satisfies

$$\bigcup_{i=1}^{3} J_i = \{1, 2, \ldots, k\}.$$

For each i, define

$$g^{(i)}(t) = \sum_{j \in J_i} a_j \chi_{(0, s_j)}(t). \tag{6}$$

Then $g^{(i)} \in \mathscr{D}$ and

$$\sum_{i=1}^{3} g^{(i)}(t) = g(t). \tag{7}$$

Moreover,

$$\begin{aligned} S_\sigma[g^{(1)}](t) &= \sum_{j \in J_1} a_j \int_0^l \chi_{(0, s_j)}(s)\, d\Psi(s, t) \\ &= \sum_{j \in J_1} a_j \Psi(s_j, t) \\ &= \sum_{j \in J_1} a_j \varphi_{X_1}(s_j)/\varphi_{Y_1}(t) \\ &= \| g^{(1)} \|_{\Lambda(X_1)} / \varphi_{Y_1}(t). \end{aligned} \tag{8}$$

The inequalities for $g^{(2)}$ and $g^{(3)}$ follow similarly.

Now suppose $g \in \mathscr{D}$ is arbitrary and let $g_m \uparrow g$ where the g_m belong to $\mathscr{D}$ and are step functions with finite support. From the first part of the proof we obtain functions $(g_m)^{(i)}$ for each m such that Eqs. (7) and (8) hold. Applying Helly's theorem, we obtain subsequences $\{(g_{m_j})^{(i)}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} (g_{m_j})^{(i)} = g^{(i)},$$

for some $g^{(i)} \in \mathscr{D}$, $(i = 1, 2, 3)$. If we let $h_j = g_{m_j}$, then by the dominated converge theorem we get

$$\lim_{j\to\infty} S_\sigma[(h_j)^{(i)}](t) = S_\sigma[g^{(i)}](t), \tag{9}$$

since $0 \leqslant (h_j)^{(i)}, g^{(i)} \leqslant g$. But Fatou's lemma for Lebesgue integration along with Eqs. (8) and (9) imply

$$\begin{aligned} \| g^{(i)} \|_{\Lambda(X_i)}/\varphi_{Y_i}(t) &\leqslant \liminf_{j\to\infty} \{\|(h_j)^{(i)}\|_{\Lambda(X_i)}/\varphi_{Y_i}(t)\} \\ &= \liminf_{j\to\infty} S_\sigma[(h_j^{(i)}](t) \\ &= S_\sigma[g^{(i)}](t). \end{aligned}$$

Relation (4) gives the opposite inequality proving the lemma.

The proof of the sufficiency now comes over verbatim from Corollary (4.4), Theorem (4.5), and Theorem (4.7) of [6] with the exception that the following inequalities are used:

$$\varphi_{Y_i}(t) \leqslant 3\varphi_{Y_i}(t/3),$$

since $\varphi_{Y_i}(t)$ is concave and

$$\sum_{i=1}^{3} T(f_i)^*(t) \leqslant \sum_{i=1}^{3} (Tf_i)^*(t/3),$$

yielding

$$(Tf)^*(t) \leqslant 3MS_\sigma[f^*](t).$$

In the general case of n pairs, the constant 3 will be replaced by n.

The theorem can be applied to the results in [6, Section 5] to obtain many specific weak interpolation theorems involving Λ_ϕ, M_ϕ, and $\Lambda_\alpha(X)$ spaces. The statements of those theorems must be modified in the obvious way.

Conjecture. One question is readily apparent, namely, can interpolation of n pairs be completely developed by appropriate iteration of interpolation of two pairs at a time? This is easily seen to have an affirmative answer if $X = X_i$ all i, or $Y = Y_i$ all i, or all fundamental functions of $X_1, Y_1, \ldots, X_n, Y_n$ are powers. We conjecture that the question has a negative answer in general.

3. Indices and Interpolation

In [3] Krein and Semenov state that inaccuracies appear in Theorem 2 of an earlier paper of the second author [5]. In this section we give a counter-example to Theorem 1 of that paper. This appears important because several interpolation theorems dealing with function spaces use this theorem (e.g., [5, 11]). We also note that the error in the theorem most likely occurs in Lemma 5 of that paper where an assumption seems to have been made that the extreme points of the unit ball of an arbitrary rearrangement invariant function space are of the form $\chi_E \operatorname{sgn} f / \| \chi_E \|_X$.

We define two types of indices for function spaces which will be important for our discussion. The fundamental indices [11] of a rearrangement invariant function space X are calculated by

$$\gamma_X = \sup_{0<s<1} \Theta_1(s) = \lim_{s\to 0} \Theta_1(s) \tag{10}$$

and

$$\bar{\gamma}_X = \inf_{1<s} \Theta_1(s) = \lim_{s\to\infty} \Theta_1(s), \tag{11}$$

where $\Theta_1(s) = \log M(s, X)/\log s$ and $M(s, X) = \sup_t\{\varphi_X(st)/\varphi_X(t)\}$.

The Boyd indices are evaluated using the norm of the dilation operator σ_a given by

$$\sigma_a[f](t) = f(at)\, \chi_{[0,l]}(t).$$

The computation of the indices is then

$$\underline{\sigma}_X = \sup_{0<s<1} \Theta(s) = \lim_{s\to 0} \Theta(s) \tag{12}$$

and

$$\bar{\sigma}_X = \inf_{s>1} \Theta(s) = \lim_{s\to\infty} \Theta(s), \tag{13}$$

where $\Theta(s) = \log h(s, X)/\log s$ and $h(s, X) = \| \sigma_{1/s} \|_X$.

Using the covariantly additive property of the norm of a Lorentz Λ_ϕ space, it is not hard to see that

$$h(a, \Lambda_\phi) = \sup_s \left\{ \int_0^{as} \phi(t)\, dt \Big/ \int_0^s \phi(t)\, dt \right\}, \tag{14}$$

where ϕ is a positive decreasing function and $\Lambda_\phi = \{f \mid \int_0^l f^* \phi\, ds < \infty\}$.

In [10] Shimogaki constructs a rearrangement invariant Banach function space X with the properties that

$$\varphi_X(t) = t^{1/2} \tag{15}$$

(and so $\underline{\gamma}_X = \bar{\gamma}_X = \frac{1}{2}$) and

$$\underline{\sigma}_X = 0. \tag{16}$$

We recount briefly the construction:

$$X = \left\{ f \mid \|f\|_X = \sup_g \int_0^1 f^* g \, ds < \infty \right\},$$

where the supremum is taken over certain fixed positive decreasing step functions W_n and K_α. It was shown that (15) was satisfied, that $\| W_n \|_X = 1 - 2^{-2n}$ and $\| \sigma_{1/n} W_n \|_X \leqslant 1 + (n-1)\, 2^{-n}$ held true. So letting $V_n = \sigma_{1/n} W_n$, he obtained

$$\| V_n \|_X \leqslant 1 + (n-1) 2^{-n}$$

and

$$\| \sigma_{1/a_n} V_n \|_X = 1 - 2^{-n},$$

when $a_n = 1/n$, so

$$b_n \leqslant \| \sigma_{1/a_n} \| = h(a_n, X),$$

where $b_n = (1 - 2^{-2n})/(1 + (n-1)\, 2^{-n})$. But b_n converges to 1 as n goes to infinity, so

$$0 \leqslant \underline{\sigma}_X = \lim_{a \to 0} \Theta(a) = \lim_{n \to \infty} \Theta(a_n) \leqslant \lim_{n \to \infty} \log(b_n)/\log a_n = 0.$$

For our purposes a separable space is needed, but unfortunately X is nonseparable. This is shown by considering the function $W = \sup_n W_n$. Then it is not too hard to see that $W \in X$ but W does not have absolutely continuous norm and hence X must be nonseparable [4]. However, if we let X_0 be the norm closure of the simple functions with finite support in X, then X_0 is a separable rearrangement invariant function space with properties (15) and (16) still holding.

The statement [5, Theorem 1] which we wish to show invalid is stated as: If X_0 is a separable rearrangement invariant function space on (0, 1) and

$$1 < \mu < \liminf_{t \to 0} \{\varphi_{X_0}(2t)/\varphi_{X_0}(t)\} \leqslant \limsup_{t \to 0} \{\varphi_{X_0}(2t)/\varphi_{X_0}(t)\} < \gamma < 2,$$

then

$$\|f\|'_{X_0} \leqslant \|f\|_{X_0} \leqslant A \, \|f\|'_{X_0}, \tag{17}$$

where

$$\|f\|'_{X_0} = \sup\{\|f\|_{\Lambda_{g^*}} \mid \|g\|_{X'_0} \leqslant 1 \text{ and } g \in \Omega(\mu, \nu)\} \tag{18}$$

and

$$\Omega(\mu, \nu) = \left\{g^* \mid \mu \leqslant \left\{\int_0^{2t} g^*(s)\, ds \Big/ \int_0^t g^*(s)\, ds\right\} \leqslant \nu, \text{ all } t\right\} \tag{19}$$

Pick μ and ν so that $1 < \mu < 2^{1/2} < \nu < 2$ and use the representation (18) to estimate $\underline{\sigma}_{X_0}$. Letting $a_n = 2^{-n}$, we have

$$\begin{aligned}
\|\sigma_{1/a_n}[f]\|_{X_0} &\leqslant A \|\sigma_{1/a_n}[f]\|'_{X_0} \\
&\leqslant A \sup\{h(a_n, \Lambda_g)\|f\|_{\Lambda_g} \mid g \in \Omega(\mu, \nu) \text{ and } \|g\|_{X'} \leqslant 1\} \\
&\leqslant A \|f\|_{X_0} \sup_g \left\{\prod_{i=1}^n \sup_t \left(\int_0^{2^{-i}t} gds \Big/ \int_0^{2(2^{-i}t)} gds\right) \Bigm| g \in \Omega(\mu, \nu), \|g\|_{X'} \leqslant 1\right\} \\
&\leqslant A\mu^{-n} \|f\|_{X_0}
\end{aligned}$$

and so

$$h(a_n, X_0) \leqslant A\mu^{-n}.$$

We then have

$$\begin{aligned}
\underline{\sigma}_{X_0} &= \lim_{a \to 0} \Theta(a) = \lim_{n \to \infty} \Theta(a_n) \\
&\geqslant \lim_{n \to \infty} \left(\frac{\log \mu}{\log 2} - \frac{\log A}{n \log 2}\right) = \frac{\log \mu}{\log 2} > 0,
\end{aligned}$$

contradicting (16).

Boyd has shown [1] that

$$1/q < \underline{\sigma}_X, \qquad \bar{\sigma}_X < 1/p, \tag{20}$$

is necessary and sufficient for (X, X) to be weak intermediate for $[(L^p, L^p), (L^q, L^q)]$ if $1 < p < q < \infty$. We need this result on several occasions in what follows.

Zippin proved in [11] a weak type interpolation theorem using a modified version of Semenov's result (17) involving fundamental indices rather than the invariants $\liminf_{t \to 0} \varphi_X(2t)/\varphi_X(t)$, etc. Zippin's result says:

If X, X_1, X_2 are separable rearrangement invariant function spaces and

$$\bar{\gamma}_{X_1} < \underline{\gamma}_X, \; \bar{\gamma}_X < \underline{\gamma}_{X_2},$$

then (X, X) is weak intermediate for $[(X_1, X_1), (X_2, X_2)]$.

We show that the space X_0 provides a counterexample to this statement with $X_1 = L^q$ and $X_2 = L^p$ where $1 < p < 2 < q < \infty$. If Zippin's result were true, then S_σ would be a bounded operator on X_0 where $\sigma = [(L^p, L^p), (L^q, L^q)]$. But notice that, in that case, S_σ automatically is bounded on X. In order to see this notice that it suffices by relation (2) to just show $S_\sigma f^*$ belongs to X for each f^* in X. Let $f_n \uparrow f^*$ where each f_n is a positive decreasing step function of finite support and therefore belongs to X_0. Since $S_\sigma[f_n] \uparrow S_\sigma[f^*]$ holds, and

$$\| S_\sigma[f_n]\|_X = \| S_\sigma[f_n]\|_{X_0} \leqslant \text{const.} \cdot \|f_n\|_{X_0} \leqslant \text{const} \|f\|_X ,$$

we have by property (v) that

$$\| S_\sigma[f^*]\|_X \leqslant \text{const} \|f\|_X .$$

The necessity of Boyd's theorem then requires, however, that $\underline{\sigma}_X > 1/q > 0$, contradicting $\underline{\sigma}_X = 0$.

We also wish to note here that Shimogaki's space X provides a counterexample to an earlier conjecture of ours on interpolation between $\Lambda(X)$ and $M(X)$. In particular, since $\varphi_X(t) = t^{1/2}$, we have

$$\Lambda(X) = L_{2,1} \hookrightarrow X \hookrightarrow L_{2,\infty} = M(X).$$

For $p < 2 < q$, S_σ maps $L_{2,r}$ continuously into $L_{2,r}$, $1 < r < \infty$ by the Stein–Weiss theorem (see also [2]). But Boyd's theorem restricts S_σ from being a bounded operator on X. Hence S_σ is an operator which is bounded on $\Lambda(X)$ and $M(X)$ but not on X itself.

In [3] the authors recognize that Zippin's result must be altered and also make the suggestion that the result holds true if X satisfies the property $\| \sigma_{1/a} \|_X \leqslant A \sup_t\{\varphi_X(at)/\varphi_X(t)\}$. Of course, this is easily recognized to imply that $\underline{\sigma}_X = \underline{\gamma}_X$ and $\bar{\sigma}_X = \bar{\gamma}_X$. We show that with this additional requirement Zippin's result is true but follows easily from the sufficiency portion of Boyd's theorem without use of the ideas and techniques set forth in [5].

We must show that S_σ is bounded on X where $\sigma = [(X_1, X_1), (X_2, X_2)]$. Pick p and q so that

$$\bar{\gamma}_{X_1} < 1/q < \underline{\sigma}_X , \bar{\sigma}_X < 1/p < \underline{\gamma}_{X_2} .$$

If we show that S_σ is of weak types (L^p, L^p) and (L^q, L^q), then the desired conclusion will follow from Boyd's theorem. Hence we only need to show that S_σ is of weak type (L^r, L^r) for each r satisfying

$$\bar{\gamma}_{X_1} < 1/r < \underline{\gamma}_{X_2} .$$

In fact, since $r > 1$ it is well known that it suffices to show that S_σ is of restricted weak type (r, r), i.e., to prove inequality (1) where $f = \chi_E$. But inequality (2) gives that it is enough to prove inequality (1) with $f = \chi_{(0,s)}$ and since $S_\sigma[\chi_{(0,s)}](t) = \Psi(s, t)$ is positive and decreasing, we only need to prove that

$$\sup_{s,t}\{\Psi(s, t)(t/s)^{1/r}\} \leqslant A. \tag{21}$$

Suppose now that $\epsilon_1 = 1/r - \bar{\gamma}_1$ (here $\bar{\gamma}_i = \bar{\gamma}_{X_i}$, $\bar{\gamma}_X = \bar{\gamma}$, etc.), then by Eq. (11) applied to both X_1 and X_2 we can find a $\delta_1 > 0$ so that

$$M(s/t, X_i) \leqslant (s/t)^{\bar{\gamma}_i+\epsilon_1}, \qquad \text{if} \quad 1/\delta_1 < s/t$$

and so

$$\Psi(s, t)(t/s)^{1/r} \leqslant \min_{i=1,2} \{(s/t)^{\bar{\gamma}_i+\epsilon_1}\} (t/s)^{1/r} = 1, \tag{22}$$

when $1/\delta_1 < s/t$.

Similarly, by setting $\epsilon_2 = \gamma_2 - 1/r$ we can use Eq. (10) applied to both X_1 and X_2 to obtain $\delta_2 > 0$ such that

$$\Psi(s, t)(t/s)^{1/r} \leqslant 1, \tag{23}$$

when $s/t < \delta_2$. Therefore letting δ be the smaller of δ_1 and δ_2, we have that inequalities (22) and (23) hold for $1/\delta < s/t$ and $s/t < \delta$, respectively. On the other hand, if $\delta \leqslant s/t \leqslant 1/\delta$, then

$$\Psi(s, t)(t/s)^{1/r} \leqslant \max_{i=1,2} \{M(1/\delta, X_i)\}\delta^{-1/r} < \infty,$$

thereby proving the assertion.

We conclude this paper by making a correction to an earlier work [7]. Page 970 line 7 f.t. should read

"... Suppose $\|f\|_X \leqslant 1$. Define $E(k) = \{x \mid |f(x)| > f^*(k)\}$ and

$$f^{(k)}(s) = \chi_{E(k)}(s) \min(|f(s)|, k) \operatorname{sgn} f(s),$$

then $m(E(k)) \leqslant k$ ..."

Page 970 inequality (3.3) should then read:

$$\begin{aligned} \| f - f^{(k)} \|_{\Lambda(X_1)+\Lambda(X_1)} &\leqslant \|(f^* - k)_+\| + \| f^*\chi_{(k,\infty)} \| \\ &\leqslant \|(1/\varphi_X - k)_+\| + \| 1/\varphi_X\, \chi_{(k,\infty)} \| \qquad (3.3) \\ &\leqslant 2 \| 1/\varphi_X - (1/\varphi_X)^{(k)} \|." \end{aligned}$$

It should also be noted that Semenov's result was used indirectly in [7] to reformulate a certain interpolation theorem involving Boyd indices into criteria involving the fundamental indices. Corollary 2 on page 979 of [7] should be changed back into the form involving the Boyd indices.

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