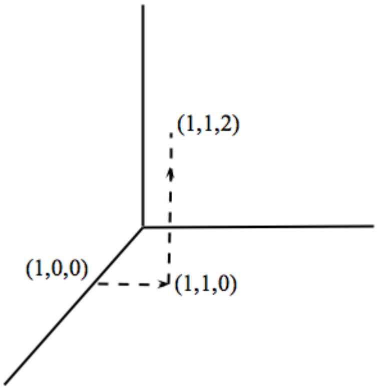
Test 2, MATH 550

Instructions: Write your name legibly. Name: ______________________________

There are 100 points. Show all work to get partial credit. You can not use calculators capable of symbolic computations, calculating derivatives or integrals.

(1) Let C be the polygonal path from $(1,0,0)$ to $(1,1,0)$ and then from $(1,1,0)$ to $(1,1,2)$ and let $\mathbf{F} = (2xy + yz)\mathbf{i} + (x^2 + xz + z^2)\mathbf{j} + (xy + 2yz)\mathbf{k}$.



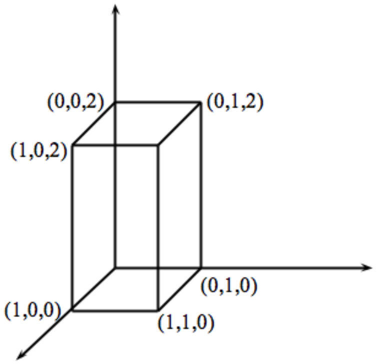
a. (8pts) Compute $\int_C \mathbf{F}\cdot d\mathbf{R}$ by parametrizing C(use 2 pieces).
Solution: $\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 F_2(1,y,0)\,dy + \int_0^2 F_3(1,1,z)\,dz = \int_0^1 1\,dy + \int_0^2 (1+2z)\,dz = 1 + 6 = 7.$

b. (8 pts)Compute that the curl $\nabla \times \mathbf{F} = \mathbf{0}$ by explicitly showing all parts.
Solution: $\nabla \times \mathbf{F} = ((x+2z) - (x+2z))\mathbf{i} - (y-y)\mathbf{j} + ((2x+z) - (2x+z))\mathbf{k} = \mathbf{0}.$

c. (8pts)Find a potential of $\mathbf{F}$.
Solution: $\phi(x,y,z) = x^2y + xyz + z^2y.$

d. (6 pts) Recompute $\int_C \mathbf{F}\cdot d\mathbf{R}$ by using this potential.
Solution: $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,1,2) - \phi(1,0,0) = (1+2+4) - 0 = 7$.

(2) Let S be the **open top** box bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, and $z=2$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + (z-x)\mathbf{k}$. Let $\mathbf{n}$ denote the unit outward normal of the box.



a. (10 points) Compute $\nabla\times\mathbf{F}$.
Solution: $\nabla\times\mathbf{F} = 0\mathbf{i} - (-1)\mathbf{j} + (1+1)\mathbf{k} = \mathbf{j} + 2\mathbf{k}$.

b. (10 pts) Compute $\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS$ by computing this integral over each of the five faces of S.
Solution: $n = \mathbf{i}$ on front, so

$$\iint_{\text{front}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{front}} 0\,dS = 0.$$

Similarly

$$\iint_{\text{back}} \nabla\times\mathbf{F}\,dS = \iint_{\text{back}} 0\,dS = 0.$$

On the bottom $\mathbf{n} = -\mathbf{k}$, so

$$\iint_{\text{bottom}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{bottom}} -2\,dS = -2.$$

On the lefts side $\mathbf{n} = -\mathbf{j}$,so

$$\iint_{\text{left}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{left}} -1\,dS = -2.$$

On the right side $\mathbf{n} = \mathbf{j}$, so

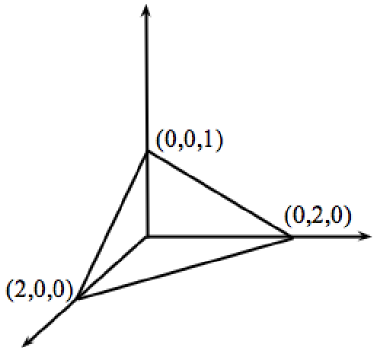
$$\iint_{\text{right}} \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{right}} 1\,dS = 2.$$

Hence $\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = -2 + (-2) + 2 = -2$.

c. (10 points) Let C be the boundary of S (i.e., the boundary of the top of the box) oriented consistently with the choice of the unit normal vectors $\mathbf{n}$. Verify Stokes' Theorem by directly computing $\int_C \mathbf{F} \cdot d\mathbf{R}$ (split into 4 pieces).

Solution: The orientation consistent with the outward normal on the sides traverses C from $(1,1,2)$ to $(1,0,2)$ to $(0,0,2)$ to $(0,1,2)$ and then back to $(1,1,2)$. Hence $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_1^0 F_2(1,y,2)\, dy + \int_1^0 F_1(x,0,2)\, dx + \int_0^1 F_2(0,y,2)\, dy + \int_0^1 F_1(x,1,2)\, dx = -1 + 0 + 0 - 1 = -2.$

(3) (10 points) Let S denote the triangle determined by the plane $x + y + 2z = 2$ and $x \geq 0$, $y \geq 0$, and $z \geq 0$.

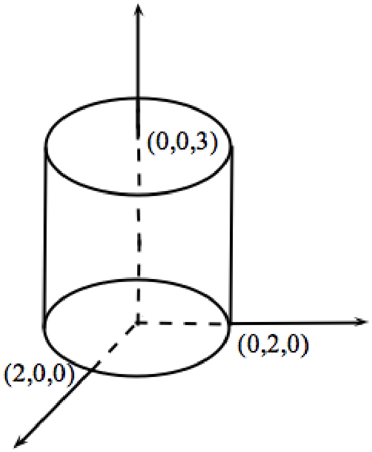


Compute the area of S by computing $\iint_S dS$.

Solution: A unit normal to S is $\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (Find the normal vector from the equation of the plane and normalize it). Thus $\cos\gamma = \mathbf{n} \cdot \mathbf{k} = \frac{2}{\sqrt{6}}$. Hence $dS = \frac{1}{|\cos\gamma|} dxdy = \frac{1}{2}\sqrt{6} dxdy$, so

$$\text{Area}(S) = \frac{1}{2}\sqrt{6} \times \text{area of triangle in } xy-\text{plane} = \sqrt{6}.$$

(4) Let S be the surface of the cylinder $x^2 + y^2 = 4$ with $0 \leq z \leq 3$ and let $\mathbf{F} = x\mathbf{i} + (y+z)\mathbf{j} - 2z\mathbf{k}$.



a. (15 pts) Compute directly $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ denotes the unit outward normal.
Solution: The unit outward normal is $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}}{2}$, so $\mathbf{F} \cdot \mathbf{n} = \frac{x^2+y^2+zy}{2} = \frac{4+zy}{2}$ on S. Also $dS = 2d\theta dz$ in cylindrical coordinates, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^3 4 + 2z \sin\theta \, dz d\theta = 24\pi.$$

b. (5 pts) Compute $\nabla \cdot \mathbf{F}$.
Solution: $\nabla \cdot \mathbf{F} = 1 + 1 - 2 = 0$.

c. (5 pts) If S_1 denotes the closed surface bounded by $z = 0$, $z = 3$, and S, compute the value of $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS$ by using the divergence theorem.
Solution: By the divergence theorem $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 0\, dV = 0$.

d. (5 pts) Use the information from **c.** to recompute $\iint_S \mathbf{F} \cdot n\, dS$ from part **a** (Compute over the top and bottom of the cylindrical region).
Solution: By part **c.** we know that $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_{\text{top}} \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{\text{bottom}} \mathbf{F} \cdot \mathbf{n}\, dS$. Now $\mathbf{n} = \mathbf{k}$ on top, so $\mathbf{F} \cdot \mathbf{k} = -2z = -6$ on top. Hence $\iint_{\text{top}} \mathbf{F} \cdot \mathbf{n}\, dS = -6 \times$ area of the top $= -6 \times 4\pi = -24\pi$. On the bottom $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F} \cdot \mathbf{n} = 2z = 0$, so $\iint_{\text{bottom}} \mathbf{F} \cdot \mathbf{n}\, dS = 0$. Hence $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -(-24\pi) - 0 = 24\pi$, which agrees with part **b.**.