Solutions Homework 9

(1) **Pg. 93: 17 Solution:**

a. Note first that $f_x(y) = a_n\chi_{[n,n+1)}(y) - a_n\chi_{[n+1,n+2)}(y)$ for $n \le x < n+1$ and $f_x(y) = 0$ for all other x. Hence f_x is integrable and obviously $\int f_x(y)\,dy = 0$ for all x. Hence $\int \left(\int f_x(y)\,dy\right)\,dx = 0$. On the other hand $f^y(x) = a_0\chi_{[0,1)}(x)$ for $0 \le y < 1$, while $f^y(x) = -a_{n-1}\chi_{[n-1,n)}(x) + a_n\chi_{[n,n+1)}(x)$ for $n \le y < n+1$. This shows that for all y the slice f^y is integrable.

b. From part **(1)** it follows that

$$\int f^y(x)\,dx = a_0\chi_{[0,1)}(y) + \sum_{n=1}^{\infty}(a_n - a_{n-1})\chi_{[n,n+1)}(y).$$

Hence $y \mapsto \int f^y(x)\,dx$ is measurable and as $a_0 + \sum_{n=1}^{\infty} a_n - a_{n-1} = \lim a_n = s < \infty$ we have that $y \mapsto \int f^y(x)\,dx$ is integrable and $\int \left(\int f^y(x)\,dx\right)\,dy = s \neq 0$.

c. Since $|f(x,y)| = \sum_{n=0}^{\infty} a_n\chi_{[n,n+1)\times[n,n+1)}(x,y) + a_n\chi_{[n,n+1)\times[n+1,n+2)}(x,y)$, it is clear that $|f|$ is a measurable function of (x,y). Form the above and Fubini's theorem it follows now that f can't be integrable. Using Tonelli's Theorem we can see that $\iint f\,dxdy = 2\sum_{n=0}^{\infty} a_n$ and this sum diverges to ∞ as the the terms do not converge to 0.

(2) **Pg. 93:19 Solution** Let $\mathcal{B} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y < |f(x)|\}$. Then as in Corollary 3.8 the set $\mathcal{B}$ is measurable and $m(\mathcal{B}) = \int m(\mathcal{B}_x)\,dx = \int |f(x)|\,dx$. On the other hand $\int m(\mathcal{B}^y)\,dy = \int_0^{\infty} m(\{x : |f(x)| > y\})\,dy$ from which the conclusion follows via Tonelli's theorem.

(3) **Pg. 93: 24**

a. Assume $|g| \le M$. let $\epsilon > 0$. Then there exists $\delta > 0$ such that $|h| < \delta$ implies $\|f - f_h\|_1 < \frac{\epsilon}{M}$ (from the previous hw). Let now $|x_1 - x_2| < \delta$. Then

$$\begin{aligned}
|(f*g)(x_1) - (f*g)(x_2)| &= \left|\int \{f(x_1 - y) - f(x_2 - y)\}g(y)\,dy\right| \\
&\le M\int |f(x_1 - y) - f(x_2 - y)|\,dy \\
&= M\int |f(x_1 - x_2 + t) - f(t)|\,dt \\
&= M\|f_{x_2 - x_1} - f\|_1 < \epsilon.
\end{aligned}$$

b. If g is also integrable, then by a theorem proved in class, we know that $f*g$ is integrable. By **a.** we know $f*g$ is also uniformly continuous. this implies, the same way as the case $d = 1$ (which was done in a previous hw) that $(f*g)(x) \to 0$ as $|x| \to \infty$.

(4) By Tonelli (or Fubini) $m(E) = \int_0^1 m(E_x)\,dx \le \frac{1}{2}$. Now by Tonelli again

$$m(\{y \in [0,1] : m(E^y) = 1\}) \le \int_0^1 m(E^y)\,dy = m(E) \le \frac{1}{2}.$$

(5) Let $g(t) = \frac{1}{2h}\chi_{[-h,h]}$. Then $g \in L^1$ and $\|g\|_1 = 1$. Now observe

$$\phi_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = g * f(x).$$

Now shown in class if $f, g \in L^1$, then $g * f \in L^1$ and $\|g * f\|_1 \leq \|g\|_1 \|f\|_1$.