Solutions for HW 4

17. Solution: We first show that there exists c_n such that

$$m\{x : \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\} < \frac{1}{2^n}.$$

Let $A_k = \{x : |f_n(x)| > k\}$. Then $m(A_k) \downarrow 0$, so there exists k_n such that $m(A_{k_n}) < \frac{1}{2^n}$. Now take $c_n = k_n n$. Then $|f_n(x)| > k_n$ is the same as $\left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}$. Let now $E_n = \{x : \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\}$. Then $\sum_n m(E_n) < \infty$. By Borel-Cantelli $m(\limsup E_n) = 0$, which says that $\frac{f_n(x)}{c_n} \to 0$ a.e.

22. Solution: Suppose first that $f, g : I \to \mathbb{R}$ are continuous functions such that $f = g$ a.e. Then $f = g$ on I. Assume $f(x_0) \neq g(x_0)$. Then by continuity $f(x) \neq g(x)$ on a delta interval around x_0, which contradicts $f = g$ a.e. Now assume there exists a continuous f such that $f = \chi_{[0,1]}$ a.e. Then the restrictions of f and $\chi_{[0,1]}$ are both continuous on $(0,1)$ and therefore $f(x) = 1$ for all $0 < x < 1$. Similarly we see that $f(x) = 0$ for $x < 0$ and $x > 1$, but this contradicts that f is continuous at $x = 0$ (and $x = 1$).

33. Solution: Assume that $m^*(\mathcal{N}^c) < 1$, Then there exists an open set U with $\mathcal{N}^c \subset U$ and $m(U) < 1$. Now $U^c \subset \mathcal{N}$ and U^c measurable, so by problem 32 (a) we have that $m(U^c) = 0$. Now $I = U \cup U^c$ implies that $1 = m(U) + m(U^c) = m(U) < 1$, which is a contradiction. Hence $m^*(\mathcal{N}^c) = 1$. Now $1 = m(I) < m^*(\mathcal{N}) + m^*(\mathcal{N}^c)$.

35. Solution: The function F is increasing and x is strictly increasing, which implies that G is strictly increasing and thus one-to-one. Now $G(0) = F(0) + 0 = 0$ and $G(1) = 1 + 1 = 2$. Hence $G([0,1]) = [0,2]$ and thus G^{-1} maps $[0,2]$ onto $[0,1]$. Now $\Phi = G^{-1}$ is strictly increasing and thus continuous. It follows that G maps disjoint open intervals onto disjoint open intervals. If $I = (a,b)$ is an open interval in$[0,1]$ which was removed to form $\mathcal{C}$, then F is constant on I, so $G(I) = F(a) + I$. Hence $m(G(I)) = m(I)$. This implies that $m(G(\mathcal{C}^c)) = 1$, as $m(\mathcal{C}^c) = 1$. This implies that $m(G(\mathcal{C})) = 2 - 1 = 1$. Let $\mathcal{N}$ be a non-measurable subset of $G(\mathcal{C})$ (exists by exercise 32). Then $E = \Phi(\mathcal{N}) \subset \mathcal{C}$, and thus E is measurable (as $m^*(E) = 0$). Now $\Phi^{-1}(E) = G(E) = \mathcal{N}$ is not measurable. Let $f = \chi_{\Phi(N)}$. Then f measurable, but $\{f \circ \Phi > 0\} = \Phi^{-1}(\Phi(\mathcal{N}) = \mathcal{N}$ is not measurable. That E is not Borel follows from the following: If f is measurable, then $f^{-1}(E)$ is measurable for all Borel sets. the proof of this fact is as follows: Let $\Lambda = \{A : f^{-1}(A) \text{ is measurable}\}$. Then Λ contains the open sets (see notes or Property 1 on page 28). In particular $\emptyset$ and $\mathbb{R}$ are in Λ. If $A \in \Lambda$, then $f^{-1}(A^c) = (f^{-1}(A))^c$ is measurable, as complements of measurable sets are measurable. Hence $A^c \in \Lambda$, whenever $A \in \Lambda$. Assume now $A_n \in \Lambda$. Then $f^{-1}(\cup_n A_n) = \cup_n f^{-1}(A_n)$ is measurable, as countable unions of measurable sets are measurable. Hence Λ is closed under countable unions. Therefore Λ is a σ-algebra, containing the open sets. By definition Λ contains all the Borel sets, i.e., $f^{-1}(B)$ is measurable for all Borel sets B.