Solutions homework 11.

(1) **Solution:** Let $0 < x \leq 1$ and $0 < \epsilon < x$. Then

$$F(x) = F(\epsilon) + \int_{\epsilon}^{x} F'(y)\, dy.$$

Now $F(\epsilon) \to F(0)$ as $\epsilon \to 0$ by the continuity of F at 0. For any sequence $\epsilon_n \to 0$ the sequence of functions $F'(y)\chi_{[\epsilon_n,x]}(y)$ converges $F'(y)\chi_{[0,x]}(y)$ a.e. and the sequence is bounded above by $|F'| \in L^1[0,1]$. Hence by the Dominated Convergence Theorem

$$\int_{\epsilon_n}^{x} F'(y)\, dy \to \int_{0}^{x} F'(y)\, dy.$$

This implies that

$$F(x) = F(0) + \int_{0}^{x} F'(y)\, dy$$

and thus F is absolutely continuous.

(2) **Solution:** As $a > 0$ we have that $|F(x)| \leq x^a \to 0$ as $x \to 0$, so F is continuous at 0. Also for $x \neq 0$ we have $F'(x) = ax^{a-1}\sin\frac{1}{x^b} - bx^{a-b-1}\cos\frac{1}{x^b}$. Hence $|F'(x)| \leq ax^{a-1} + bx^{a-b-1} \in L^1[0,1]$, so $F' \in L^1[0,1]$. For $\epsilon > 0$ the function F' is continuous on $[\epsilon, 1]$, and thus bounded. This implies that F is Lipschitz on $[\epsilon, 1]$ and thus absolutely continuous on $[\epsilon, 1]$. From problem 2 it follows that F is absolutely continuous and thus of bounded variation on $[0,1]$.

(3) **Solution:**

a. Let $A = \{x \in (0,1) : f'(x) = 1\}$. Then A measurable since f' measurable and $f(x) = f(0) + \int_0^x f'(t)\, dt = m(A \cap (0,x))$, so **b.** holds.

b. If $f(x) = m(A \cap (0,x))$, then $f(x) = \int_0^x \chi_A(t)\, dt$ implies that f is absolutely continuous. By the second fundamental theorem of calculus $f'(x) = \chi_A(x)$ a.e., so **a.** holds.

(4) **Solution:** Since f_n is absolutely continuous and $f_n(0) = 0$ we have $f_n(x) = \int_0^x f_n'(t)\, dt$. Hence

$$|f_n(x) - f_m(x)| \leq \int_0^1 |f_n'(t) - f_m'(t)|\, dt \to 0$$

uniformly in x as $n, m \to \infty$. Thus there exists $f : [0,1] \to \mathbb{R}$ such that f_n converges uniformly to f. To prove f is absolutely continuous, observe that $\{f_n'\}$ is a Cauchy sequence in $L^1([0,1])$. Hence there exists $g \in L^1([0,1])$ such that $\|f_n' - g\|_1 \to 0$. This implies that

$$\int_0^x f_n'(t)\, dt \to \int_0^x g(t)\, dt$$

as $n \to \infty$ for every $x \in [0,1]$ (even uniformly in x). Hence $f(x) = \int_0^x g(t)\, dt$, which implies that f is absolutely continuous.

(5) **Solution:** Let $\epsilon > 0$. Then there exists $\delta > 0$ such that if $\{(c_i, d_i)\}$ is finite disjoint collection of open intervals in $[c,d]$ with $\sum(d_i - c_i) < \delta$, then

$$\sum |g(d_i) - g(c_i)| < \epsilon.$$

Then there exists $\delta_1 > 0$ such that if $\{(a_i, b_i)\}$ is finite disjoint collection of open intervals in $[a,b]$ with $\sum(b_i - a_i) < \delta_1$, then

$$\sum |f(b_i) - f(a_i)| < \delta.$$

Now $\{(f(a_i), f(b_i))\}$ is finite disjoint collection in $[c, d]$, since f is increasing. Hence

$$\sum |g(f(b_i)) - g(f(a_i))| < \epsilon.$$