Solutions for HW 1

Pg. 37: 1 Solution: Let $x, y \in \mathcal{C}$ such that $x \neq y$. Then $|x-y| > 0$, so there exists a k such that $|x-y| > \frac{1}{3^k}$, since $\frac{1}{3^k} \to 0$ as $k \to \infty$. Then x, y belong to two different closed subintervals of $\mathcal{C}_k$. Hence there is a non-empty open interval between x and y which is disjoint from the Cantor set, so $\mathcal{C}$ is totally disconnected. To prove $\mathcal{C}$ is perfect let $x \in \mathcal{C}$. We will show that x is not isolated. For each $k \geq 1$ there is a closed interval J_k in $\mathcal{C}_k$ such that $x \in J_k$ and the endpoints of J_k are in $\mathcal{C}$. Let y_k be an endpoint of $\mathcal{C}_k$ with $x \neq y_k$ (at least one of the two endpoints works). Then $y_k \to x$ implies that x is a limit point of $\mathcal{C}$.

Pg. 38: 2. Solution

a. First let x be such that $x = \sum_{n=1}^{m} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$. We prove by induction on m that in that case x is a left endpoint of an interval in $\mathcal{C}_m$. If $m = 1$, then $x = 0$ or $x = \frac{2}{3}$, so the claim is then true. Suppose now that we have proved this for $m-1$. Let $x = \sum_{n=1}^{m} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$. Then $\sum_{n=1}^{m-1} \frac{a_n}{3^n}$ is by assumption a left endpoint of an interval in $\mathcal{C}_{m-1}$. If $a_m = 0$, then x is the left endpoint of the left subinterval of that interval and if $a_m = 2$ then x is the left endpoint of the right subinterval of that interval, so by induction we have proved our claim about these special x's. Now let x be such that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. Then we have

$$\sum_{n=1}^{m} \frac{a_n}{3^n} \leq x \leq \sum_{n=1}^{m} \frac{a_n}{3^n} + \sum_{n=m+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{m} \frac{a_n}{3^n} + \frac{1}{3^m}.$$

If now $\sum_{n=1}^{m} \frac{a_n}{3^n}$ is the left endpoint of the closed interval J_m in $\mathcal{C}_m$, then the above inequalities show that $x \in J_m \in \mathcal{C}_m$. Hence $x \in \mathcal{C}_m$ for all m, i.e., $x \in \mathcal{C}$. Assume now $x \in \mathcal{C} = \cap_{k=1}^{\infty} \mathcal{C}_k$. Then $x \in \mathcal{C}_1$, so define $a_1 = 0$ in case $0 \leq x \leq \frac{1}{3}$ and define $a_1 = 2$ in case $\frac{2}{3} \leq x \leq 1$. Assume now that $a_1, \ldots, a_{m-1}$ have been defined, so that if $x \in J_{m-1}$ where J_{m-1} is one of the closed subintervals of $\mathcal{C}_{m-1}$, then $\sum_{n=1}^{m-1} \frac{a_n}{3^n} \in J_{m-1}$. Now x is in the left subinterval of J_{m-1} or in the right subinterval of J_{m-1} In the first case define $a_m = 0$ and in the second case define $a_m = 2$. Proceeding this way we obtain a_m such that $|x - \sum_{n=1}^{m} \frac{a_n}{3^n}| \leq \frac{1}{3^m}$. This shows that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_n \in \{0.2\}$.

b. Let $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and $y = \sum_{n=1}^{\infty} \frac{a'_n}{3^n}$, where $a_n, a'_n \in \{0.2\}$. Assume $|x - y| < \frac{1}{3^k}$. Then x, y belong to the same closed subinterval of $\mathcal{C}_k$ and therefore $a_j = a'_j$ for $j = 1, \cdots, k$. In particular the expansion of $x \in \mathcal{C}$ is unique, when we use the representation with 0's and 2's. This implies immediately that F is well-defined on $\mathcal{C}$. To prove continuity let $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where $a_n \in \{0.2\}$ and $\epsilon > 0$ Then there exists a k such that $\frac{1}{2^k} < \epsilon$. Then for $y = \sum_{n=1}^{\infty} \frac{a'_n}{3^n}$, where $a'_n \in \{0, 2\}$ such that $|x - y| < \frac{1}{3^k}$ we have $|F(x) - F(y)| < \frac{1}{2^k} < \epsilon$. Hence F is continuous on $\mathcal{C}$. It is obvious that $F(0) = 0$ and $F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

c. Let $0 \le y \le 1$. Then y has a binary expansion $y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$ (if y has more than one expansion, choose either one). Then define $x = \sum_{k=1}^{\infty} \frac{2b_k}{3^k}$. Then $x \in \mathcal{C}$ and $F(x) = y$.

Note: One can use this to give a different proof of the continuity of F. Part (c) shows that F is onto $[0,1]$ so that if we show that F is increasing then F is automatically continuous (as a discontinuity would be a jump discontinuity which would make the range disconnected). To prove F increasing one needs to show that if $x < y$ and $x, y \in \mathcal{C}$, then the ternary expansions of x and y agree up to an index k, while the $k+1$ term is 0 for x and 2 for y. This implies that $F(x) < F(y)$.

d. If (a,b) is an interval removed from $[0,1]$ to form $\mathcal{C}_m$, then the right endpoint b is the left endpoint of a closed interval of $\mathcal{C}_m$. Hence $b = \sum_{k=1}^{m} \frac{a_k}{3^k}$ with $a_m = 2$. Then $a = b - \frac{1}{3^k} = \sum_{k=1}^{m-1} \frac{a_k}{3^k} + \sum_{k=m+1} \frac{2}{3^k}$. From this it follows easily that $F(a) = F(b)$. Define $F(x) = F(a)$ on (a,b). Then $F : [0,1] \to [0,1]$ and F is automatically continuous on $[0,1] \setminus \mathcal{C}$, since that set is open. Let $x \in \mathcal{C}$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that $|x - y| < \delta$ and $y \in \mathcal{C}$ implies that $|F(x) - F(y)| < \epsilon$. Assume now $|x - y| < \delta$ with $y \in [0,1] \setminus \mathcal{C}$. Then $y \in (a,b)$ for some interval removed to form $\mathcal{C}$. Then either $|x - a| < \delta$ or $|x - b| < \delta$. Assume wlog $|x - a| < \delta$. Then $a \in \mathcal{C}$ implies $|F(x) - F(a)| < \epsilon$, but $F(y) = F(a)$ so also $|F(x) - F(y)| < \epsilon$. This proves that F is continuous on $[0,1]$.

Pg. 41 Problem 12 Solution

a. Let B be any open disc in $\mathbb{R}^2$ and let R be an open rectangle contained in B. Then the boundary of R can have at most four points in common with the boundary of B, so there exist a point on the boundary of R which is not on the boundary of B. If B is a disjoint union of open rectangles, then such a boundary point would have to be an interior point of another rectangle, which is impossible as every disc around such a point must intersect the first rectangle.

b. Assume O is open and connected and a disjoint union of two or more open rectangles. Let $x, y \in O$ belonging two distinct rectangles R_1, respectively R_2. Then O is path-wise connected, so there exists a continuous piecewise linear function $f : [a,b] \to O$ such that $f(a) = x$ and $f(b) = y$. Let $t_0 = \sup\{t : f(t) \in R_1\}$. Then $f(t_0) \notin R_1$, as R_1 is open, but then $f(t_0)$ is on the boundary of R_1 and we get a contradiction the same way as in part (a).

One can argue also as follows: If $O = \cup_i R_i$ is a disjoint union of open rectangles containing more than one rectangle, let R_{i_0} be one of them. Then $O = R_{i_0} \cup (\cup_{i \neq i_0} R_i)$ is a union of two disjoint open sets, which is a contradiction.