A SIMPLE COMPLEX ANALYSIS AND AN ADVANCED CALCULUS PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

ANTON R. SCHEP

In this note we present two proofs of the Fundamental Theorem of Algebra. The first one uses Cauchy's integral form and seems not to have been observed before in the literature. The second one, which uses only results from advanced calculus, is the real variable version of the complex analysis proof. This proof was motivated by the proof of [1], where the same ideas were used to prove a more general result (the non-emptiness of the spectrum of an element in a complex Banach algebra).

Theorem (Fundamental Theorem of Algebra). *Every polynomial of degree $n \geq 1$ with complex coefficients has a zero in $\mathbb{C}$.*

Proof. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ and assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$.
First Proof: By Cauchy's integral theorem we have

$$\oint_{|z|=r} \frac{1}{zp(z)}\, dz = \frac{2\pi i}{p(0)} \neq 0,$$

where the circle is traversed counter clockwise. On the other hand $|p(z)| = |z|^n|(1 + \frac{a_1}{z} + \cdots + \frac{a_0}{z^n})| \to \infty$ as $|z| \to \infty$ implies that

$$\left| \oint_{|z|=r} \frac{1}{zp(z)}\, dz \right| \leq 2\pi \max_{|z|=r} \frac{1}{|p(z)|} \to 0,$$

as $|z| \to \infty$, which is a contradiction.
Second Proof Define $g : [0, \infty) \times [0, 2\pi] \to \mathbb{C}$ by $g(r, \theta) = \frac{1}{p(re^{i\theta})}$. Then the function g is continuous on $[0, \infty) \times [0, 2\pi]$ and has continuous partials on $(0, \infty) \times (0, 2\pi)$, given by

$$\frac{\partial g}{\partial r}(r, \theta) = \frac{-e^{i\theta}}{p^2(re^{i\theta})}$$
$$\frac{\partial g}{\partial \theta}(r, \theta) = \frac{-rie^{i\theta}}{p^2(re^{i\theta})}.$$

Define now $F : [0, \infty) \to \mathbb{C}$ by $F(r) = \int_0^{2\pi} g(r, \theta)\, d\theta$. Then F is continuous on $[0, \infty)$ by the uniform continuity of g on $[0, M] \times [0, 2\pi]$ and by Leibniz's rule for differentiation under the integral sign we have for all $r > 0$

$$F'(r) = \int_0^{2\pi} \frac{\partial g}{\partial r}(r, \theta)\, d\theta = \int_0^{2\pi} \frac{-e^{i\theta}}{p^2(re^{i\theta})}\, d\theta,$$

Date: May 4, 2007.

so that by the fundamental theorem of calculus

$$riF'(r) = \int_0^{2\pi} \frac{-rie^{i\theta}}{p^2(re^{i\theta})}\,d\theta = \int_0^{2\pi} \frac{\partial g}{\partial\theta}(r,\theta)\,d\theta = g(r,2\pi) - g(r,0) = 0.$$

Hence $F'(r) = 0$ for all $r > 0$. This implies that F is constant on $[0,\infty)$ with $F(r) = F(0) = \frac{2\pi}{p(0)} \neq 0$. On the other hand $|p(z)| \to \infty$ as $|z| \to \infty$ implies that $g(r,\theta) \to 0$ as $r \to \infty$ uniformly in θ. Therefore $F(r) \to 0$ as $r \to \infty$, which is a contradiction. $\square$

References

[1] Dinesh Singh, The spectrum in a Banach algebra, *The Amer. Math. Monthly* **113**(2006) 756–758.

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
E-mail address: `schep@math.sc.edu`